PY4126 Radiative Processes

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1 Introduction

These rough lecture notes for PY4126 Radiative processes. This part of the course consists of 18 of the 24 lectures and is based on [1], with some additions from [2, 3, 4]. The standard reference for all things electrodynamics is [5], at a couple of points we will encounter arguments from here but for the most part it is a bit more advanced than the level of this course. Sections with a $*$ next to them were not lectured and a included for completeness.

1.1 Notation and units conventions

Throughout the course we take the convention of denoting vector quantities with an overhead arrow, e.g. $\vec{r}$ for the position vector. We also follow the conventions of [2] and denote the Cartesian basis vectors as $\hat{x}, \hat{y}, \hat{z}$ to bring them in line with the convention for other coordinate systems. This means that in Cartesian coordinates the position vector is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. In Appendix C we review some of the necessary vector calculus for the course and make some comments on the coordinate systems that we need to be aware of.

In this course we follow [2] and use SI , or mks, units. Many books, including [1] and [5] use Gaussian units, sometimes known as cgs or centimetres-grams-seconds. Gaussian are probably still the standard unit system for books and courses in electrodynamics. However, many people now follow the example of [2] and teach them exclusively in SI units. There is an appendix in [2] which explains how to transform between the two systems I replicate some of that here so that you can have an easier time of comparing to some of the other books. The main reason for working in Gaussian units is that it simplifies some expressions such as Coulomb's law:

$$F_C = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) \quad \text{in Gaussian,} \tag{1.1}$$

$$F_C = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) \quad \text{in SI.} \tag{1.2}$$

In [2] The notation $\mathscr{r} = \vec{r}_1 - \vec{r}_2$ is used for the distance between a source point $\vec{r}_2$ and a field point $\vec{r}_1$. We will not use that notation but will sometimes write $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ for the relative position vector.

1.2 Correspondence between lectures and these notes

Lectures 6,7 and 8 covered the material in Section. 2. Lectures 9 - 12 covered the material in Section. 3, the start of lecture 22 covered the material on radiation reaction. Lectures 13, 14, 17 and 18 covered Section. 4.1. The end of lecture 14 and lecture 15 covered Section. 4.2. Lecture 16 covered the material on polarisation in Section. 2. Lectures 19, 20 and the first part of 21 covered Section. 5 on propagation of em waves through a plasma. Finally, most of lecture 21 sketched out absorption and emission following Section. 6.

2 Basic Theory of Radiation Fields

2.1 Maxwell's equations and electromagnetic flux

For the most part this course studies the electromagnetic properties of non-relativistic particles. The central objects are thus the electric and magnetic fields, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ respectively. They are observed through their action on a particle of charge q, in other words through the Lorentz force:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{2.1}$$

The factor of the speed of light in the magnetic field term is due to working in Gaussian units. The rate of work done by the fields on a particle of charge q is

$$\vec{v} \cdot \vec{F} = q\vec{v} \cdot \vec{E}, \tag{2.2}$$

since $\vec{v} \cdot \left(\vec{v} \times \vec{B}\right) = 0$. We can write this in terms of the mechanical work for a non relativistic particle. Using Newton's Second law $F = m\frac{d\vec{v}}{dt}$ we have

$$q\vec{v} \cdot \vec{E} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}^2\right) = \frac{d}{dt}\left(U_{\text{mech}}\right). \tag{2.3}$$

From the Lorentz force law we see that the total force density, the force per unit volume, is

$$\mathcal{F} = \rho\vec{E} + \vec{J} \times \vec{B}. \tag{2.4}$$

The two new quantities ρ and $\vec{J}$ are the charge and current densities respectively. They are usually defined as

$$Q = \sum_i q_i = \int \rho dV, \tag{2.5}$$

$$Q = \int_{t_1}^{t_2} \int_\Sigma \vec{J} \cdot \hat{n} dA dt, \tag{2.6}$$

we could also write $\vec{J} = \rho\vec{v}$. In [1] the inverse relations are given:

$$\rho = \lim_{\Delta V \to 0} \frac{Q}{\Delta V}, \tag{2.7}$$

$$\vec{J} = \lim_{\Delta V \to 0} \frac{\sum_i q_i \vec{v}_i}{\Delta V}, \tag{2.8}$$

with ΔV a volume element. Here the sum is over the constituent particles of charge q_i moving with velocity $\vec{v}_i$. The rate of work done by the field per unit volume is then

$$\frac{1}{\Delta V}\sum_i q_i \vec{v}_i \cdot \vec{E} = \vec{J} \cdot \vec{E}. \tag{2.9}$$

This is nothing but the rate of change of mechanical energy from Eq. (2.3),

$$\frac{dU_{\text{mech}}}{dt} = \vec{J} \cdot \vec{E}. \tag{2.10}$$

The fundamental equations of electromagnetism are Maxwell's equations. Written in their differential form they relate $\vec{E}, \vec{B}, \rho$ and $\vec{J}$. In SI units they are

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho, \qquad \nabla \cdot \vec{B} = 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \end{aligned} \tag{2.11}$$

The vector fields $\vec{D}$ and $\vec{H}$ are the electric and magnetic "excitation" as they are induced by the $\vec{E}$ and $\vec{B}$ fields in a medium. They are related through the constitutive relations

$$\vec{D} = \varepsilon \vec{E}, \tag{2.12}$$

$$\vec{B} = \mu \vec{H}, \tag{2.13}$$

where ε, μ are respectively the dielectric constant (permittivity) and magnetic permeability of the medium[2]. In the absence of dielectric or permeable media $\varepsilon = \varepsilon_0, \mu = \mu_0$. In older texts the fields have different names as summarised in Table 1.

Field	Traditional name	More "modern" name
$\vec{E}$	Electric field strength	Electric field strength
$\vec{B}$	Magnetic induction	Magnetic field strength
$\vec{D}$	Electric displacement	Electric excitation
$\vec{H}$	Magnetic field strength	Magnetic excitation

Table 1: Traditional and more modern names for the four fields that appear in Maxwell's equations.

Sometimes ρ and $\vec{J}$ are given the subscript f to signify that they are the charge density and the current density due to *free* charges. This is to give the distinction between charges that are able to move freely through the medium and the charges that are part of the medium. Maxwell's equations can be alternatively written as

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho, \qquad \nabla \cdot \vec{B} = 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \end{aligned} \tag{2.14}$$

where now ρ and $\vec{J}$ refer to all of the charges and currents present, not just those that are free.

[2] We can write these as $\varepsilon = \epsilon_0 \epsilon_r$, $\mu = \mu_0 \mu_r$. Here ϵ_0, μ_0 are the permittivity and permeability of free space, or vacuum, while the symbols with the label r are due to the medium. Recall that ε_0 and μ_0 are related to the speed of light as $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. The label r stands for relative, and these quanitites are related to the electric and magnetic susceptibility through $\chi_e = \varepsilon_r - 1$, $\chi_m = \mu_r - 1$.

A direct consequence of Maxwell's equations is the conservation of electric charge, also known as the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \tag{2.15}$$

This follows by taking the divergence of the equation for $\vec{H}$, and recalling that the divergence of a curl vanishes, $\nabla \cdot \left(\nabla \times \vec{H}\right) = 0$.

The energy density and the energy flux, or Poynting vector, can now be defined. Start from the work done per unit volume and apply Maxwell's equations

$$\begin{aligned}
\vec{J} \cdot \vec{E} &= \left(\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{E}, \quad \text{rewriting of Maxwell's equation for } \vec{H} \\
&= \left(\vec{E} \cdot \left(\nabla \times \vec{H}\right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) \\
&= \left(\vec{H} \cdot \left(\nabla \times \vec{E}\right) - \nabla \cdot \left(\vec{E} \times \vec{H}\right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) \\
&= \left(-\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \nabla \cdot \left(\vec{E} \times \vec{H}\right) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right), \quad \text{using Maxwell's equation for } \vec{E} \\
&= \left(-\frac{\vec{B}}{\mu} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \varepsilon \vec{E}}{\partial t} - \nabla \cdot \left(\vec{E} \times \vec{H}\right)\right).
\end{aligned}$$

Going between the second and third line we used the vector identity

$$\nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\nabla \times \vec{E}\right) - \vec{E} \cdot \left(\nabla \times \vec{H}\right). \tag{2.16}$$

Finally if ε, μ are constant in time we arrive at

$$-\nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{J} \cdot \vec{E} + \frac{1}{2}\frac{\partial}{\partial t}\left[\varepsilon \vec{E}^2 + \frac{\vec{B}^2}{\mu}\right]. \tag{2.17}$$

This is known as Poynting's theorem in differential form. In words it says that: the rate of change of mechanical energy density, $\vec{J} \cdot \vec{E}$, plus the rate of change of field energy density, $\frac{1}{2}\frac{\partial}{\partial t}\left[\varepsilon \vec{E}^2 + \frac{\vec{B}^2}{\mu}\right]$, equals the negative divergence of the field energy flux, $-\nabla \cdot \left(\vec{E} \times \vec{H}\right)$. The field energy flux is known as the Poynting vector,

$$\vec{S} = \vec{E} \times \vec{H}. \tag{2.18}$$

This is actually one of four possible Poynting vectors, depending on the context and how we split the energy up between the matter energy density and the field energy density the other choices can become relevant. The electromagnetic field energy density is written as

$$U_{field} = \frac{1}{2}\left[\varepsilon \vec{E}^2 + \frac{\vec{B}^2}{\mu}\right] = U_E + U_B, \tag{2.19}$$

it contains information about the matter the field is interacting with through ε and μ. If all of the matter contribution is accounted for in the mechanical energy density then we only see the fields $\vec{E}$ and $\vec{B}$, it is like we are working in vacuum. In this case $\vec{J} \to$ conduction current + induced molecular current, and $\vec{S} \to \frac{1}{\mu_0}\vec{E} \times \vec{B}$ one of the other choices for the Poynting vector. This is the approach taken in Chapter 8 of [2]. This grouping of the energy into matter and field contributions is arbitrary, it is the total energy that is conserved!

There is an integral form of Poynting's theorem found by integrating Eq. (2.17) and using the Divergence theorem:

$$\begin{aligned} -\int_\Sigma \vec{S} \cdot d\vec{A} &= -\int_V \nabla \cdot \vec{S}\, dV \\ &= \int_V \vec{J} \cdot \vec{E}\, dV + \frac{d}{dt}\int_V \left(\frac{\varepsilon \vec{E}^2 + \frac{\vec{B}^2}{\mu}}{2}\right) dV, \end{aligned}$$

where Σ is the surface which forms the boundary to the volume V. This gives the relation

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left(U_{\text{mech}} + U_{\text{field}}\right) = -\int_\Sigma \vec{S} \cdot d\vec{A}. \tag{2.20}$$

Again this tells us that the rate of change of the energy density is given by the flux through the surface at the boundary.

The microscopic momentum density $\vec{g}$ and angular momentum density, $\vec{\mathcal{L}}$, are related to the flux density and are given by

$$\vec{g} = \frac{1}{c}\vec{E} \times \vec{B} \qquad \vec{\mathcal{L}} = \vec{r} \times \vec{g}. \tag{2.21}$$

We have been talking about charges in a volume in three dimensions $V \subset \mathbb{R}^3$, with boundary the surface Σ. The prototypical example is to consider a the volume as the interior of a ball. Then $\Sigma = S^2$ is the boundary, the surface of the ball known as the two-sphere. What happens to the flux $-\int_\Sigma \vec{S} \cdot d\vec{A}$ when the radius of the ball becomes very large, $r \to \infty$?

Recall from earlier E&M that for static situations the electric and magnetic fields have the asymptotic behaviour

$$\lim_{r\to\infty} \vec{E}, \vec{B} \sim \frac{1}{r^2}, \tag{2.22}$$

thus the Poynting vector behaves as

$$\lim_{r\to\infty} \vec{S} \sim \frac{1}{r^4} \tag{2.23}$$

and the flux vanishes in the large r limit. When the fields are time dependent this is no longer the case. Time dependent $\vec{E}, \vec{B}$ fields can fall off as $\frac{1}{r}$ which implies that $\lim_{r\to\infty} \vec{S} \sim \frac{1}{r^2}$ leading to non zero flux. This contribution of the flux to $\frac{dE}{dt}$ is due to radiation. It is conventional to refer to the $\frac{1}{r}$ decaying pieces of the $\vec{E}$ and $\vec{B}$ fields as the radiation field.

2.2 Planar electromagnetic waves

The vacuum Maxwell's equations are

$$\begin{aligned} \nabla \cdot \vec{E} &= 0, \qquad \nabla \cdot \vec{B} = 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{2.24}$$

These are invariant under the change[3] $\left(\vec{E}, \vec{B}\right) \mapsto \left(\vec{B}, -\vec{E}\right)$.

In this case Maxwell's equations can be reduced to two wave equations, leading to travelling wave solutions for the fields. To do this consider the curl of the third of Maxwell's equations. The left hand side becomes: $\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$, while the right hand side is $-\frac{\partial \nabla \times \vec{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$. Putting this together leads to

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \tag{2.25}$$

the analogous equation holds for $\vec{B}$,

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2},. \tag{2.26}$$

In Cartesian coordinates[4] these have plane wave solutions

$$\vec{E} = \hat{a}_1 E_0 e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}, \tag{2.27}$$

$$\vec{B} = \hat{a}_2 B_0 e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}, \tag{2.28}$$

with: E_0, B_0 complex constants, $\hat{a}_1$ and $\hat{a}_2$ unit vectors giving the direction of oscillation, and $\vec{k}, \omega$ the wave vector and the frequency. The wave vector can also be written in terms of the wave number, k and the direction of propagation, $\hat{n}$, as $\vec{k} = k\hat{n}$. As the equations are linear we can consider superpositions of plane waves to construct more general solutions. Substituting the plane waves into Maxwell's equations they are reduced to

$$\vec{k} \cdot \hat{a}_1 E_0 = 0, \qquad \vec{k} \cdot \hat{a}_2 B_0 = 0, \tag{2.29}$$

$$\vec{k} \times \hat{a}_1 E_0 = \omega \hat{a}_2 B_0, \quad \vec{k} \times \hat{a}_2 B_0 = -\frac{\omega}{c^2}\hat{a}_1 E_0. \tag{2.30}$$

A consequence of Eq. (2.29) is that $\hat{a}_1$ and $\hat{a}_2$ are transverse to the direction of propagation $\vec{k}$. Then Eq. (2.30) implies that $\hat{a}_1, \hat{a}_2, \hat{n}$ form a right handed bases, in other words $\hat{n} \times \hat{a}_1 = \hat{a}_2$

[3]This is known as electromagnetic duality and is a big topic in its own right. For us it just reflects that fact that $\vec{E}$ and $\vec{B}$ have the same functional form and are just vector fields propagating in the same direction and oscillating with the same amplitude.

[4]In Cartesian coordinates $\nabla^2 \vec{E} = \Delta E_x \hat{x} + \Delta E_y \hat{y} + \Delta E_z \hat{z}$ with $\Delta = \nabla \cdot \nabla$ the Laplacian. This equation for $\vec{E}$ then becomes a wave equation in each component. For other coordinate systems it is not so easy to make sense of $\nabla^2 \vec{E}$, in fact it is defined through the identity $\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$ and called the vector Laplacian.

and its cyclic permutations hold. These properties imply that $E_0 = \frac{\omega}{k} B_0$ and $B_0 = \frac{\omega}{c^2 k} E_0$ which combine to give $E_0 = \left(\frac{\omega}{kc}\right)^2 E_0$. This implies that $\omega^2 = c^2 k^2$ or, if k, ω are positive,

$$\omega = ck. \tag{2.31}$$

This relationship between the frequency and the wave number is known as the dispersion relation. There are two immediate consequences of this analysis. The first is that $E_0 = B_0$. The second is that the phase velocity of the wave propagation is

$$v_{\text{ph}} = \frac{\omega}{k} = c \tag{2.32}$$

so the waves travel at the speed of light. We can also compute the group velocity and find that it is also the speed of light, $v_{\text{gr}} = \frac{d\omega}{dk} = c$.

As both $\vec{E}$ and $\vec{B}$ vary sinusoidally in time, the Poynting vector and the energy density also fluctuate. The way to deal with this is to work with the time average of quantities. Using the result of Problem B.1 that

$$\langle \text{Re}(A)\text{Re}(B) \rangle = \frac{1}{2}\text{Re}\left(\mathcal{A}\mathcal{B}^*\right), \tag{2.33}$$

and recalling that $\vec{E}$ and $\vec{B}$ are real vector fields we find

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0}\text{Re}(E_0 B_0^*)\hat{n} = \frac{|E_0|^2}{2c\mu_0}\hat{n} = \frac{c}{2\mu_0}|B_0|^2\hat{n}. \tag{2.34}$$

Similarly the field energy density satisfies

$$\langle U_{\text{field}} \rangle = \frac{1}{4}\left(\epsilon_0 |E_0|^2 + \mu_0 |B_0|^2\right) = \frac{1}{2}\epsilon_0 |E_0|^2 = \frac{1}{2}\frac{|B_0|^2}{\mu_0}, \tag{2.35}$$

where we use that $B_0 = \frac{1}{c}E_0$ and that $c^2 = \epsilon_0 \mu_0$. This implies that the velocity of the energy flow, the ratio of flux to field energy density, is

$$\frac{\langle S \rangle}{\langle U_{\text{field}} \rangle} = \frac{1}{c\varepsilon_0 \mu_0} = c, \tag{2.36}$$

where $\langle S \rangle = \frac{1}{2c\mu_0}|E_0|^2$ is the time average of the magnitude of the Poynting vector.

These results are for the propagation of electromagnetic waves in vacuum. Formally the same results hold in a medium, e.g. when $\mu, \varepsilon \neq 1$ but are still constant. In real materials μ, ε are modified in response to the presence of the electromagnetic fields and depend on the frequency of the field, ω. This means that care is needed. We may return to this when we discuss plasmas later in the course.

2.3 Polarisation and Stokes parameters

The plane wave solutions in Eq. (2.27) and Eq. (2.28) describe monochromatic (one frequency) plane waves. They are called linearly polarised, with $\hat{a}_1, \hat{a}_2$ sometimes called the polarisation vector of the field. This is because $\vec{E}$ oscillates in the $\hat{a}_1$ direction while propagating in the $\vec{k}$ direction. The plane defined by $\hat{a}_1, \vec{k}$ is known as the plane of polarisation.

A more general description of polarisation follows from considering the superposition of plane waves (with the same frequency),

$$\vec{E} = (\hat{x}E_1 + \hat{y}E_2)\, e^{-i\omega t}, \tag{2.37}$$

where we are considering the field at a single spatial point and have decomposed $\hat{a}_1 E$ in terms of $\hat{x}E_1 + \hat{y}E_2$. Here $E_1 = \mathcal{E}_1 e^{i\phi_1}, E_2 = \mathcal{E}_2 e^{i\phi_2}$ are the modulus argument decomposition of the prefactors. As usual the physical field is the real part of $\vec{E}$, this has components

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1)\,, \tag{2.38}$$
$$E_y = \mathcal{E}_2 \cos(\omega t - \phi_2)\,. \tag{2.39}$$

The components thus oscillate with different amplitudes and frequencies[5]. In the $\hat{x}, \hat{y}$ plane the tip of $\vec{E}$ traces out an ellipse. Thus the field is called elliptically polarised.

The general equation for an ellipse with principal axes x', y' and semi-major, semi minor axes a, b is

$$\left(\frac{E_{x'}}{a}\right)^2 + \left(\frac{E_{y'}}{b}\right)^2 = 1. \tag{2.40}$$

This is solved by letting

$$\left(\frac{E_{x'}}{a}\right)^2 = \cos^2(\omega t)\,, \tag{2.41}$$
$$\left(\frac{E_{y'}}{b}\right)^2 = \sin^2(\omega t)\,. \tag{2.42}$$

We thus have a choice when taking the square root as to whether we take $E_{x'}, E_{y'}$ to be the positive or negative square root. If we make the same choice for both we find that $\vec{E}$ traces out the ellipse counter clockwise,

$$E_{x'} = a\cos(\omega t)\,, \tag{2.43}$$
$$E_{y'} = b\sin(\omega t)\,, \tag{2.44}$$

if we take the opposite sign square root then $\vec{E}'$ traces out the ellipse clockwise,

$$E_{x'} = a\cos(\omega t)\,, \tag{2.45}$$
$$E_{y'} = -b\sin(\omega t)\,. \tag{2.46}$$

Include figure of an ellipse.

Decomposing a and b in terms of a radius and an angle as $a = \mathcal{E}_0 \cos\beta$, $b = \mathcal{E}_0 \sin\beta$. The standard convention is that when $0 < \beta < \frac{\pi}{2}$ $\vec{E}'$ rotates clockwise as t increases:

$$E_{x'} = \mathcal{E}_0 \cos\beta \cos(\omega t)\,, \tag{2.47}$$
$$E_{y'} = -\mathcal{E}_0 \sin\beta \sin(\omega t)\,, \tag{2.48}$$

[5]This can become more complicated if $\mathcal{E}$ and ϕ depend on time. The fields would no longer be elliptically polarised and are called partially elliptically polarised.

and counter clockwise when $-\frac{\pi}{2} < \beta < 0$:

$$E_{x'} = \mathcal{E}_0 \cos\beta \cos(\omega t), \tag{2.49}$$
$$E_{y'} = \mathcal{E}_0 \sin\beta \sin(\omega t). \tag{2.50}$$

The case of clockwise rotation is called right handed elliptic polarisation, and the counter clockwise case is called left handed elliptic polarisation.

Special cases: There are a few special cases that we need to be aware of.

- $\beta = \pm\frac{\pi}{4}$. This implies that $\sin\beta = \mp\frac{1}{\sqrt{2}}$ and $\cos\beta = \frac{1}{\sqrt{2}}$, components are

$$E_{x'} = \frac{\mathcal{E}_0}{\sqrt{2}} \cos(\omega t), \tag{2.51}$$
$$E_{y'} = \mp\frac{\mathcal{E}_0}{\sqrt{2}} \sin(\omega t). \tag{2.52}$$

  This traces out a circle and the wave is called circularly polarised.

- $\beta = 0$. $\sin\beta = 0$ and $\cos\beta = 1$ so that $E_{y'} = 0$ and $E_{x'} = \mathcal{E}_0 \cos\omega t$ giving a linearly polarised wave.

- $\beta = \pm\frac{\pi}{2}$. This is again linearly polarised, since $\sin\beta = \mp 1$,$\cos\beta = 0$ which implies that $E_{y'} = \mp\mathcal{E}_0 \sin\omega t$ and $E_{x'} = 0$.

If we consider a more general situation where the principle axes of the ellipse are tilted by an angle χ relative to the axes x and y. Include figure of rotated ellipse. We thus have that

$$E_x = E_{x'} \cos\chi - E_{y'} \sin\chi = \mathcal{E}_0 (\cos\beta \cos\chi \cos\omega t + \sin\beta \sin\chi \sin\omega t), \tag{2.53}$$
$$E_y = E_{y'} \cos\chi + E_{x'} \sin\chi = \mathcal{E}_0 (\cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t). \tag{2.54}$$

Comparison with the expressions in Eqs. (2.38) and (2.39) we have that:

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi, \tag{2.55}$$
$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi, \tag{2.56}$$
$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi, \tag{2.57}$$
$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi. \tag{2.58}$$

The parameters $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$ describe the physical fields, while $\mathcal{E}_0, \beta, \chi$ describe the ellipse, and hence the polarisation.

It is convenient to introduce the Stokes parameters

$$I = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2, \tag{2.59}$$
$$Q = \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi, \tag{2.60}$$
$$U = 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi, \tag{2.61}$$
$$V = 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta. \tag{2.62}$$

In terms of the STokes parameters we have the following quantities: the intensity

$$I = \mathcal{E}_0^2, \tag{2.63}$$

the degree of circular polarisation

$$\frac{V}{I} = \sin 2\beta, \tag{2.64}$$

and the angle of linear polarisation χ

$$\frac{U}{Q} = \tan 2\chi. \tag{2.65}$$

Returning to the special cases from above we see that

- For $\beta = 0, \pm\frac{\pi}{2}$ we have $\sin 2\beta = 0$ so $V = 0$. The case of purely linear polarisation.
- For $\beta = \pm\frac{\pi}{4}$ $\sin 2\beta = \pm 1$ and $V = \pm I$, the case of pure circular polarisation.

For elliptic polarisation the Stokes parameters are related through

$$I^2 = Q^2 + U^2 + V^2. \tag{2.66}$$

For the partially elliptically polarised case, mentioned above when $\mathcal{E}$ and ϕ are time dependent, they satisfy

$$I^2 > Q^2 + U^2 + V^2. \tag{2.67}$$

The degree of polarisation, the percentage of the wave that is elliptically polarised, is defined as

$$\Pi = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}. \tag{2.68}$$

The wave is unpolarised when $Q = U = V = 0$. There is more on this in [1].

2.4 Electromagnetic potentials

Considering the microscopic form of Maxwell's equations, Eq. (2.14) for $\vec{E}$ and $\vec{B}$, we can express the electric and magnetic fields in terms of a scalar and vector potential, $\phi(\vec{r}, t), \vec{A}(\vec{r}, t)$. This gives a simpler formalism where we have a general method to solve Maxwell's equations.

Recall that the divergence of a curl is zero[6] so $\nabla \cdot \vec{B} = 0$ implies that $\vec{B} = \nabla \times \vec{A}$, for some vector field $\vec{A}$ called the vector potential. The Equation for $\nabla \times \vec{E}$ then becomes

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \tag{2.69}$$

[6] The converse is true in flat space, if we were working on a sphere or a torus then while $\nabla \cdot \left(\nabla \times \vec{A} \right) = 0$, $\nabla \cdot \vec{B} = 0$ does not imply that $\vec{B}$ is the curl of some vector potential $\vec{A}$.

which implies that $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\phi$, recall that the curl of a gradient is zero. This means that in terms of ϕ and $\vec{A}$ the electric and magnetic field are given by

$$\vec{B} = \nabla \times \vec{A}, \tag{2.70}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}. \tag{2.71}$$

This expression for the fields in terms of the potentials automatically solve two of Maxwell's equations. What are the consequences of the other two? Consider

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot \vec{E} = -\left[\nabla^2\phi + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right)\right], \tag{2.72}$$

and

$$\begin{aligned}
\mu_0 \vec{J} &= \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\
&= \nabla \times \left(\nabla \times \vec{A}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right) \\
&= -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right)
\end{aligned}$$

where in the last line the identity $\nabla \times \left(\nabla \times \vec{A}\right) = -\nabla^2\vec{A} + \nabla\left(\nabla \cdot \vec{A}\right)$. The potentials are not uniquely determined, they can be modified by an arbitrary scalar field in such a way that the fields $\vec{E}$ and $\vec{B}$ are left unchanged:

$$\vec{A} \mapsto \vec{A} + \nabla\psi \tag{2.73}$$

$$\phi \mapsto \phi - \frac{\partial \psi}{\partial t} \tag{2.74}$$

and $\left(\vec{E}, \vec{B}\right) \mapsto \left(\vec{E}, \vec{B}\right)$. The transformations in Eq. (2.73) and Eq. (2.74) are called gauge transformations and electromagnetism is called a gauge theory. Gauge theories show up in many areas of physics and Maxwell's theory of electromagnetism is the prototypical example.

As ψ is arbitrary it can be chosen in such a way that the equations for $\vec{A}$, ϕ simplify. In other words by choosing ψ we can specify an equation that $\vec{A}, \phi$ solve. Two of the most common examples are the Coulomb gauge $\nabla \cdot \vec{A} = 0$ and the Lorentz gauge

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{2.75}$$

Applying the Lorentz gauge condition the equations for $\vec{A}, \phi$ become

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}, \tag{2.76}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{J}. \tag{2.77}$$

These are both wave equations with a source term on the right hand side. In fact often they are written in terms of the d'Alembert operator $\Box\phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi$. These equations can be solved in terms of the Green's functions of the differential operator[7], the solution $G(\vec{r}-\vec{r}', t-t')$ to $\Box G(\vec{r}-\vec{r}', t-t') = -4\pi\delta(\vec{r}-\vec{r}', t-t')$. The solutions are given by the retarded potentials

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|} d^3\vec{r'}, \tag{2.78}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{A}(\vec{r'},t_r)}{|\vec{r}-\vec{r'}|} d^3\vec{r'}, \tag{2.79}$$

where $t_r = t - \frac{1}{c}|\vec{r}-\vec{r'}|$ is the retarded time. In other words the field at position $\vec{r}$ at time t depends on the source at position $\vec{r'}$ at time t_r. With $t - t_r$ the time that it takes light to travel between $\vec{r}$ and $\vec{r'}$.

The method to solve Maxwell's equations is the following: pick a gauge for $\phi, \vec{A}$, solve for the retarded potentials using the Green's function method, then construct $\vec{E}$ and $\vec{B}$ from the potentials. This approach is useful in many areas of physics when we need to solve a differential equation with a source term.

3 Radiation from Moving Charges

3.1 Liénard–Wiechert potentials

In the previous section we saw how to solve Maxwell's equations in terms of the scalar and vector potentials. Now we want to use the retarded potentials, Eq. 2.78 and Eq. (2.79), to find the radiation component of the fields for a moving charg.

To achieve this goal consider a charge, q, moving along the trajectory $\vec{r} = \vec{r}_0(t)$ with velocity $\vec{u}(t) = \dot{\vec{r}}_0(t)$. The charge and current densities are

$$\rho(\vec{r},t) = q\delta^3\left(\vec{r}-\vec{r}_0(t)\right), \tag{3.1}$$

$$\vec{J}(\vec{r},t) = q\vec{u}(t)\delta^3\left(\vec{r}-\vec{r}_0(t)\right), \tag{3.2}$$

where $\delta^3\left(\vec{r}-\vec{r}_0(t)\right)$, is the three dimensional Dirac delta function. The presence of the delta function localises the charge and current to the trajectory of the particle[8]. The charge and

[7] The Green's function is $G(\vec{r}-\vec{r}', t-t') = \frac{\delta\left(t'-t+\frac{|\vec{r}-\vec{r'}|}{c}\right)}{|\vec{r}-\vec{r'}|}$. The solution ϕ is then given by integrating the source times the Green's function over all space and time, $\phi(\vec{r},t) = \int\left(\int \rho\left(\vec{r'},t'\right) G(\vec{r}-\vec{r}', t-t')dt'\right) d^3\vec{r'}$. In the main text we have carried out the time integral which the Dirac delta function localises to the retarded time $t_r = t - \frac{|\vec{r}-\vec{r'}|}{c}$. In Section. 3 we will use the full expression and perform the $\vec{r'}$ integral first.

[8] These are sometimes known as the world line charge and current densities, this is because they are localised to the world line of the charge. If you have not come across the concept of a world line in special relativity it is just a fancy name for the trajectory of a particle in space-time.

current are found by integrating the densities over a three dimensional volume:

$$q = \int \rho(\vec{r},t) d^3\vec{r}, \tag{3.3}$$

$$q\vec{u} = \int \vec{J}(\vec{r},t) d^3\vec{r}. \tag{3.4}$$

We can use Eq. (3.1) and Eq. (3.2) to compute the retarded potentials due to the moving charge. Once we know the potentials we can then find the electric and magnetic fields due to the charge and identify the radiation component. In this section we will write the expressions for the potentials in terms of the Green's function of the d'Alembert operator

$$G(\vec{r}-\vec{r}', t-t') = \frac{\delta\left(t'-t+\frac{|\vec{r}-\vec{r'}|}{c}\right)}{|\vec{r}-\vec{r'}|} \tag{3.5}$$

The scalar potential is

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\left(\int \rho\left(\vec{r'},t'\right)\frac{\delta\left(t'-t+\frac{|\vec{r}-\vec{r'}|}{c}\right)}{|\vec{r}-\vec{r'}|}dt'\right)d^3\vec{r'}, \tag{3.6}$$

using $\rho(\vec{r},t) = q\delta^3\left(\vec{r}-\vec{r}_0(t)\right)$ the $\vec{r}'$ integral is localised to $\vec{r}_0(t')$,

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta\left(t'-t_r\right)}{|\vec{r}-\vec{r}_0(t')|}dt' \tag{3.7}$$

Before performing this integral it is convenient to introduce some notation:

$$\vec{R}(t') = \vec{r}-\vec{r}_0(t'), \tag{3.8}$$

$$R(t') = |\vec{R}(t')|. \tag{3.9}$$

Using this notation leads to

$$\phi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int \frac{\delta\left(t'-t+\frac{R(t')}{c}\right)}{R(t')}dt', \tag{3.10}$$

$$\vec{A}(\vec{r},t) = \frac{q\mu_0}{4\pi}\int u(t')\frac{\delta\left(t'-t+\frac{R(t')}{c}\right)}{R(t')}dt'. \tag{3.11}$$

These integrals can be massaged a bit further to make them as simple as possible to carry out. The argument of the delta function vanishes when $t'-t+\frac{R(t')}{c}=0$ this is when $t'=t_r$ the retarded time. This can be rewritten as $R(t_r) = c(t-t_r)$. Now make the substitution $t'' = t'-t+\frac{R(t')}{c}$ which implies that the $dt'' = \left(1+\frac{\dot{R}(t')}{c}\right)dt'$. Differentiating $R^2(t') = \vec{R}(t')\cdot\vec{R}(t')$ leads to $R(t')\dot{R}(t') = -\vec{R}(t')\cdot\vec{u}(t')$ since $\vec{u}(t') = \dot{\vec{r}}_0(t') = -\frac{d}{dt'}\left(\vec{r}-\vec{r}_0(t')\right)$. Next introduce the unit vector $\hat{n} = \frac{\vec{R}}{r}$. Putting this together the measure becomes

$$\begin{aligned} dt'' &= \left(1 + \frac{\dot{R}(t')}{c}\right) dt' \\ &= \left(1 - \frac{\vec{R}(t')}{R} \cdot \vec{u}(t')\right) dt' \\ &= \left(1 - \frac{1}{c}\hat{n}(t') \cdot \vec{u}(t')\right) dt' \\ &=: \kappa(t')dt'. \end{aligned}$$

This leads to the expression for the scalar potential being

$$\phi(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \int \delta(t'') \frac{dt''}{R(t')\kappa(t')}, \tag{3.12}$$

the delta function localises this to $t'' = 0$, which is equivalent to $t' = t_r$. Thus the scalar potential is

$$\phi(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R(t_r)\kappa(t_R)}. \tag{3.13}$$

A similar computation leads to

$$\vec{A}(\vec{r}, t) = \frac{q\mu_0}{4\pi} \frac{\vec{u}(t_r)}{R(t_r)\kappa(t_R)} = \frac{\vec{u}}{c^2}\phi(\vec{r}, t). \tag{3.14}$$

These are known as the Liénard–Wiechert potentials. The fact that the charge is moving has two important consequences in contrast to the static case:

1. The factor $\kappa = \left(1 - \frac{1}{c}\hat{n}(t') \cdot \vec{u}(t')\right)$ does not appear in the static case. It becomes more important the nearer the particle's velocity is to the speed of light c.

2. The terms in the potential are all evaluated at t_r rather than at t.

The finiteness of the speed of light is the reason that the potential is evaluated at t_r. This is because the values of the potential and field at position $\vec{r}$ at time t depend on what the charge is doing at position $\vec{r}_0(t_r)$ since it takes light $t - t_r$ to travel between the two positions. As t_r has an implicit r dependence care needs to be taken when differentiating $\phi, \vec{A}$ to compute the electric and magnetic fields.

3.2 Radiation fields

Identifying the radiation component of the field involves a length computation carried out in both [2, 5]. Some of the details are worth seeing so a brief discussion of the calculation is presented here.

Again imagine a charge q moving with velocity $\vec{u} = \dot{\vec{r}}_0(t_r)$ and acceleration $\dot{\vec{u}} = \ddot{\vec{r}}_0(t_r)$. Let $\vec{\beta} = \frac{\vec{u}}{c}$ which means that $\kappa = 1 - \hat{n} \cdot \vec{\beta}$. The key feature is that t_r is differentiated:

$$\nabla t_r = -\frac{1}{c\kappa}\hat{n}, \tag{3.15}$$

$$\frac{\partial t_r}{\partial t} = \frac{1}{\kappa}. \tag{3.16}$$

Checking this is a worthwhile endeavour for interested student. Armed with these derivatives we can compute the gradient of Eq. (3.13) and the time derivative of Eq. (3.14). Starting with ϕ:

$$\begin{aligned}
\nabla\phi &= -\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\nabla\left(R\kappa\right), \\
&= -\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\nabla\left(R - \vec{R}\cdot\vec{\beta}\right), \\
&= -\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\left(-c\nabla t_r - \nabla\left(\vec{R}\cdot\vec{\beta}\right)\right), \qquad \text{using } R = c\left(t - t_r\right).
\end{aligned}$$

Next use the product rule for the gradient of a scalar product,

$$\nabla\left(\vec{R}\cdot\vec{\beta}\right) = \left(\vec{R}\cdot\nabla\right)\vec{\beta} + \left(\vec{\beta}\cdot\nabla\right)\vec{R} + \vec{R}\times\left(\nabla\times\vec{\beta}\right) + \vec{\beta}\times\left(\nabla\times\vec{R}\right). \tag{3.17}$$

Computing the four terms on the right hand side gives:

$$\left(\vec{R}\cdot\nabla\right)\vec{\beta} = \dot{\vec{\beta}}\left(\vec{R}\cdot\nabla t_r\right), \tag{3.18}$$

$$\left(\vec{\beta}\cdot\nabla\right)\vec{R} = \vec{\beta} - c\vec{\beta}\left(\vec{\beta}\cdot\nabla t_r\right), \tag{3.19}$$

$$\vec{R}\times\left(\nabla\times\vec{\beta}\right) = -\dot{\vec{\beta}}\left(\vec{R}\cdot\nabla t_r\right) + \nabla t_r\left(\dot{\vec{\beta}}\cdot\vec{R}\right), \tag{3.20}$$

$$\vec{\beta}\times\left(\nabla\times\vec{R}\right) = c\vec{\beta}\left(\vec{\beta}\cdot\nabla t_r\right) - c\nabla t_r\vec{\beta}^2, \tag{3.21}$$

again the details are left as an exercise. Adding the four terms together leads to

$$\nabla\left(\vec{R}\cdot\vec{\beta}\right) = \vec{\beta} + \nabla t_r\left(\dot{\vec{\beta}}\cdot\vec{R} - c\beta^2\right), \tag{3.22}$$

substituting this in to the gradient of the scalar potential gives

$$\nabla\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\left[\vec{\beta} + c\nabla t_r\left(1 - \beta^2 + \dot{\vec{\beta}}\cdot\frac{\vec{R}}{c}\right)\right]. \tag{3.23}$$

Using Eq. (3.15) this becomes

$$\nabla\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\left[\vec{\beta} - \frac{1}{\kappa}\hat{n}\left(1 - \beta^2 + \dot{\vec{\beta}}\cdot\hat{n}\frac{R}{c}\right)\right]. \tag{3.24}$$

A similar computation, making use of Eq. (3.16) and $\mu_0 = \frac{1}{c^2\varepsilon_0}$ leads to

$$\frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^2}\left[\frac{R}{c}\dot{\vec{\beta}} - \vec{\beta} + \frac{1}{\kappa}\left(1 - \beta^2 + \frac{R}{c}\hat{n}\cdot\dot{\vec{\beta}}\right)\vec{\beta}\right]. \tag{3.25}$$

The electric field is given by

$$\begin{aligned}\vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \\ &= -\frac{q}{4\pi\varepsilon_0}\frac{1}{R^2\kappa^3}\left[\left(\vec{\beta} - \hat{n}\right)(1-\beta^2) - \frac{R}{c}\left(-\kappa\dot{\vec{\beta}} + \left(\hat{n} - \vec{\beta}\right)\left(\dot{\vec{\beta}}\cdot\hat{n}\right)\right)\right],\end{aligned}$$

the second term is neatened up by using

$$\hat{n}\times\left(\left(\hat{n} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right) = \left(\hat{n} - \vec{\beta}\right)\left(\hat{n}\cdot\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\kappa, \tag{3.26}$$

leading to

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\left(\hat{n} - \vec{\beta}\right)(1-\beta^2)}{R^2\kappa^3} + \frac{q}{4\pi\varepsilon_0 c}\frac{\left[\hat{n}\times\left(\left(\hat{n} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right)\right]}{R\kappa^3}. \tag{3.27}$$

The first term in Eq. (3.27) falls of as $\frac{1}{R^2}$ and is know as the velocity field, while the second term falls of as $\frac{1}{R}$ is called the radiation or acceleration field,

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c}\frac{\left[\hat{n}\times\left(\left(\hat{n} - \vec{\beta}\right)\times\dot{\vec{\beta}}\right)\right]}{R\kappa^3}. \tag{3.28}$$

A very similar computation, see Problem. B.5, gives the magnetic field due to a moving charge,

$$\vec{B} = \frac{\hat{n}}{c}\times\vec{E}, \tag{3.29}$$

with the radiation component

$$\vec{B}_{\text{rad}} = \frac{\hat{n}}{c}\times\vec{E}_{\text{rad}}. \tag{3.30}$$

3.3 Radiation from Non-Relativistic Particles

This section could also be titled "A derivation of the Larmor formula" as this is the expression which governs the power radiated by a point charge. Throughout we will work with charges moving in vacuum, to get the results in a medium requires changing $\mu_0, \varepsilon_0 \to \mu, \varepsilon$ and keeping track of any subtleties that this introduces. In Equations (3.27) and (3.29) we know what the fields of an accelerating charged particle is. To compute the power radiated we need to the Poynting vector

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{1}{\mu c}\left(\vec{E}\times\left(\hat{n}\times\vec{E}\right)\right) = \frac{1}{\mu c}\left(E^2\hat{n} - \left(\hat{n}\cdot\vec{E}\right)\vec{E}\right). \tag{3.31}$$

To find $P = \oint \vec{S}\cdot d\vec{a}$ we integrate over a two sphere of radius R with $R\to\infty$, recalling that the power radiated depends on the fields at the retarded time t_r, see Figure. 1 for a sketch of what this looks like.

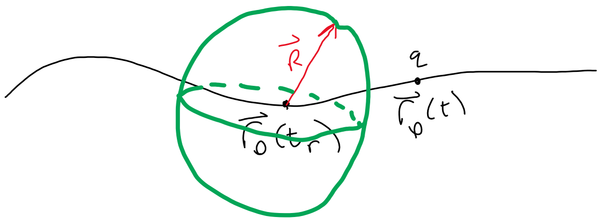

Figure 1: A moving charge radiates in a sphere.

As was discussed above upon integration only the radiation field survives, as the velocity field gives terms proportional to $\frac{1}{R^4}$ which go to zero as $R \to \infty$. Also from Equation. (3.28) we know that $\vec{E}_{\text{rad}}$ is perpendicular to $\hat{n}$ which means that

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}|E_{\text{rad}}|^2\hat{n}. \tag{3.32}$$

As a first attempt let us consider the non-relativistic case, that is where the velocity of the particle is much smaller than the speed of light ($\beta \ll 1$ or $v \ll c$ which implies that $\kappa \simeq 1$). If $\hat{n}$ and $\dot{\vec{\beta}}$ have an angle θ between them then,

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\hat{n} \times (\hat{n} \times \vec{a})}{R} = \frac{\mu_0 q}{4\pi R}\left[(\hat{n} \cdot \vec{a})\,\hat{n} - \vec{a}\right]. \tag{3.33}$$

Thus the Poynting vector is

$$\vec{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi R}\right)^2\left(a^2 - (\vec{a} \cdot \hat{n})^2\right)\hat{n} = \frac{\mu_0 q^2 a^2}{16\pi^2 R^2}\sin^2\theta\,\hat{n}. \tag{3.34}$$

This leads to radiation in a doughnut around the acceleration vector, as $\theta = 0$ for $\hat{n}$ parallel (or anti-parallel) to $\vec{a}$ there is no radiation in the direction of acceleration. Figure 2, taken from [2], shows the shape of this radiation profile.

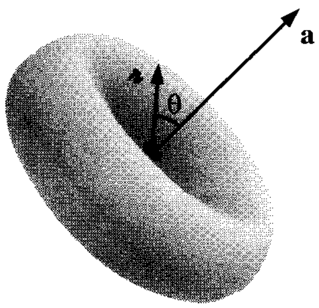

Figure 2: Radiation from an accelerating charge occurs in a doughnut.

To find the total power radiated we just need to integrate the Poynting vector over the sphere of radius R:

$$P = \oint \vec{S}_{\text{rad}} \cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^{2\pi} \int_0^{\pi} \frac{\sin^2\theta}{R^2} R^2 \sin\theta d\theta d\phi = \frac{\mu_0 q^2 a^2}{6\pi c}, \tag{3.35}$$

this is known as the Larmor formula. In carrying out the integrals we used that $\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}$. In [2] a relativistic generalisation of this is results is given, i.e where $v \ll c$ is no longer assumed,

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) \tag{3.36}$$

with $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. This is known as Liénard's generalisation and reduces to Equation (3.35) when $v \ll c$. The γ^6 means that the radiated power increases drastically as the particles velocity approaches the speed of light.

An important observation from Equation (3.35) is that the power is proportional to the square of the charge and the square of acceleration. This means that particles radiate the same power regardless of if they are positively or negatively charged, it also means that only accelerating particles radiate. Finally, it implies that the radiation is the same independent of if the particle is accelerating or decelerating.

Example 3.1. Special case : radiating dipole
Consider a charge q dipole that is oscillating along the z-axis. The dipole moment is given by $\vec{d} = q\vec{r}$, for position vector $\vec{r}$. Accounting for the oscillation this is given by

$$\vec{d} = d_0 \cos(\omega t)\,\hat{e}_z = \Re\left\{d_0 e^{i\omega t}\hat{e}_z\right\}. \tag{3.37}$$

In complex notation this means that

$$q\dot{v} = \ddot{\vec{d}} = -\omega d_0 e^{-i\omega t}\hat{e}_z = -\omega^2\vec{d}. \tag{3.38}$$

The electric radiation field is then given by

$$\begin{aligned}\vec{E}_{\text{rad},d} &= \frac{\mu_0 q\dot{v}}{4\pi r}\sin\theta\hat{e}_\theta \\ &= -\frac{\mu_0\omega^2 d_0}{4\pi r}\sin\theta\hat{e}_\theta\Re\left\{e^{-i\omega t_r}\right\} \\ &= -\frac{\mu_0\omega^2 d_0}{4\pi r}\sin\theta\hat{e}_\theta\Re\left\{e^{-i(\omega t - kr)}\right\},\end{aligned} \tag{3.39}$$

using this the time average of the Poynting vector is

$$\langle S_{\text{rad}}\rangle = \frac{\varepsilon_0 c}{2}|E|^2 = \frac{\mu_0}{32\pi^2 c r^2}\omega^4|d_0|^2\sin^2\theta, \tag{3.40}$$

and the time average of the power radiated is

$$\langle P_{\text{rad}}\rangle = \frac{\mu_0}{12\pi c}\omega^4|d_0|^2. \tag{3.41}$$

This is the same result as the Larmor formula, the factor of $\frac{1}{2}$ difference is due to the time averaging.

3.4 Radiation from Relativistic Particles

What about radiation from particles moving at relativistic velocities? We could start from Equation (3.36) and derive the results of the following section, instead we will take a slightly different approach. The acceleration can be split into two pieces $\vec{a} = \vec{a}_{\text{par}} + \vec{a}_{\text{perp}}$, parallel and perpendicular to the velocity. We will treat these two cases separately.

3.4.1 Parallel Radiation

Case I: $\vec{a}$ parallel to $\vec{v}$. In this case $\vec{\beta} \times \vec{a}$=0 so the radiation fields become:

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c} \frac{\hat{n}\left(\hat{n} \times \dot{\vec{\beta}}\right)}{R\kappa^3} = \frac{\mu_0 q}{4\pi} \frac{\hat{n}\,(\hat{n} \times \vec{a})}{R\kappa^3}, \tag{3.42}$$

$$\vec{B}_{\text{rad}} = \frac{\hat{n}}{c} \times \vec{E}_{\text{rad}} = \frac{\mu_0 q}{4\pi c R\kappa^3} \left[\hat{n} \times (\hat{n} \times (\hat{n} \times \vec{a}))\right] = \frac{\mu_0 q}{4\pi c} \frac{\vec{a} \times \hat{n}}{R\kappa^3}. \tag{3.43}$$

In finding $\vec{B}_{\text{rad}}$ we have expanded the vector triple product and used that $\hat{n} \times (\hat{n} \times \vec{a}) = 0$. We also have that $\kappa = 1 - \hat{n} \cdot \vec{\beta} = 1 - \beta \cos\theta$. This gives a $\frac{1}{(1-\beta\cos\theta)^3}$ term which changes the direction that the radiation is emitted in, tilting it towards the velocity. Pictorially, the doughnut of radiation is pushed forward and stretched out, see Figure 3 for a schematic of what happens.

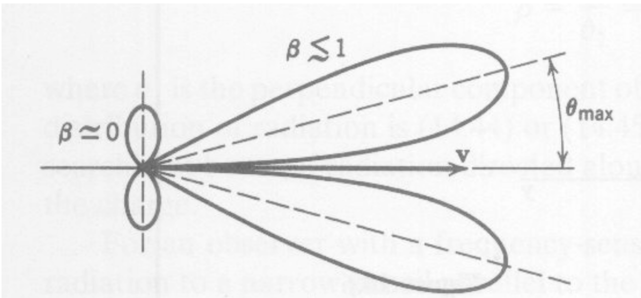

Figure 3: Radiation from a point charges whose acceleration is parallel to its velocity. The doughnut of Figure 2 has been pushed forward and stretched out. This picture is rotationaly symmetric around $\vec{v}$.

A natural next step is to consider the θ-dependence of the emitted radiation and calculate the angle of maximum emission, θ_{max}. Before doing this recall that the emitted power **is not** equal to the detected power. This is due to the build up of wave fronts in front of a moving charge. The $\vec{E}_{\text{rad}}, \vec{B}_{\text{rad}}$ fields above give the detected fields rather than the emitted fields. In terms of the work done on the charges the detected power is

$$P_{\text{rad}}(t) = -\frac{dW}{dt}. \tag{3.44}$$

Since the particle is moving, the change in the retarded time, t_r, is different than the change in detector time, t. What we want is

$$P_{\text{rad}}(t_r) = -\frac{dW}{dt_r}. \tag{3.45}$$

From Equation (3.16) it follows that $\frac{\partial f}{\partial t_r} = \kappa \frac{\partial f}{\partial t}$. This leads to

$$P_{\text{rad}}(t_r) = -\frac{dW}{dt_r} = -\kappa \frac{dW}{dt} = \kappa P_{\text{rad}}(t). \tag{3.46}$$

If we consider the angular dependence on a sphere of radius R, $\frac{dP_{\text{rad}}(t_r)}{d\Omega} = -\frac{d^2W}{d\Omega dt_r}$ with solid angle $d\Omega$, we find the emission profile,

$$\begin{aligned} -\frac{d^2W}{d\Omega dt_r} &= -\kappa \frac{d^2W}{d\Omega dt} = \kappa |\vec{S}_{\text{rad}}| R^2 \\ &= \frac{\kappa}{\mu_0 c} \left|E_{\text{rad}}\right|^2 R^2 \\ &= \frac{\kappa}{\mu_0 c} \left(\frac{\mu_0 q}{4\pi}\right)^2 \frac{\left|\hat{n} \times (\hat{n} \times \vec{a})\right|^2}{R^2 \kappa^6} \\ &= \frac{\mu_0 q^2 |\vec{a}|^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}. \end{aligned}$$

The angular dependence of the emitted power is thus

$$\frac{dP_{\text{rad}}(t_r)}{d\Omega} = \frac{\mu_0 q^2 |\vec{a}|^2}{16\pi^2 c} \frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5}. \tag{3.47}$$

To find the angle at which the emitted power is maximum consider $\frac{d}{d\theta}\frac{dP_{\text{rad}}(t_r)}{d\Omega} = 0$. A fairly staight forward computation, see Problem B.6, results in

$$\cos\theta_{\text{max}} = \frac{\sqrt{1+15\beta^2}-1}{3\beta} \tag{3.48}$$

For strongly relativistic motion with $\beta \simeq 1$, e.g. $\gamma \gg 1$, this becomes

$$\cos\theta_{\text{max}} = \frac{1}{2}\frac{1}{\gamma}. \tag{3.49}$$

The total power emitted is found by integrating this over the solid angle, $d\Omega = \sin\theta d\theta d\varphi = d\left(\cos\theta\right) d\varphi$. This leads to

$$P_{\text{rad}}(t_r) = \int \frac{dP_{\text{rad}}(t_r)}{d\Omega} d\Omega \tag{3.50}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\sin^2\theta}{\left(1-\beta\cos\theta\right)^5} d\left(\cos\theta\right) \tag{3.51}$$

$$= \frac{\mu_0 q^2 a^2}{16\pi^2 c} 2\pi \frac{4}{3} \frac{1}{\left(1-\beta^2\right)^3} \tag{3.52}$$

$$= \frac{\mu_0 q^2 a^2}{6\pi c}\gamma^6. \tag{3.53}$$

The first part, $\frac{\mu_0 q^2 a^2}{6\pi c}$, is the Larmor formula of the previous section with γ^6 the relativistic correction.

Notice that the angular distribution is the same for both accelerating and decelerating charges. The radiation from a rapidly decelerating charge is called Bremsstrahlung, or braking radiation. We will return to this in Section 4.2.

3.4.2 Perpendicular Radiation

Case II: $\vec{a}$ perpendicular to $\vec{v}$. When the acceleration is perpendicular to the velocity there are two angles in the problem; α between $\hat{n}$ and the velocity, and θ between $\hat{n}$ and the acceleration, as shown in Figure 4. This means that

$$\vec{\beta} \cdot \hat{n} = \beta \cos \alpha, \quad \vec{a} \cdot \hat{n} = a \cos \theta. \tag{3.54}$$

The vector term in $\vec{E}_{\text{rad}}$, Equation (3.28), becomes

$$\left(\hat{n} \times \left(\left(\hat{n} - \vec{\beta}\right) \times \vec{a}\right)\right) = \left(\hat{n} - \vec{\beta}\right)(\hat{n} \cdot \vec{a}) - \vec{a}\left(\hat{n} \cdot \left(\hat{n} - \vec{\beta}\right)\right) \tag{3.55}$$

$$= \left(\hat{n} - \vec{\beta}\right) a \cos \theta - \vec{a}\,(1 - \beta \cos \alpha) \tag{3.56}$$

and note that

$$\left(\hat{n} - \vec{\beta}\right)^2 = 1 - 2\beta \cos \alpha + \beta^2. \tag{3.57}$$

The modulus of the Poynting vector is $|\vec{S}_{\text{rad}}| = \varepsilon_0 c |E_{\text{rad}}|^2$, squaring $\left(\hat{n} \times \left(\left(\hat{n} - \vec{\beta}\right) \times \vec{a}\right)\right)$

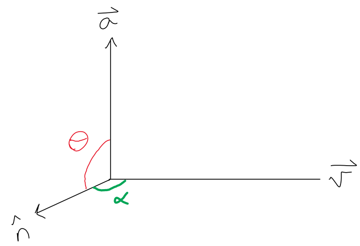

Figure 4: When the acceleration is perpendicular to the velocity we have two angles; α between $\hat{n}$ and the velocity, and θ between $\hat{n}$ and the acceleration.

gives

$$\left(\hat{n} \times \left(\left(\hat{n} - \vec{\beta}\right) \times \vec{a}\right)\right) = \left(\hat{n} - \vec{\beta}\right)^2 a^2 \cos^2 \theta - 2\vec{a} \cdot \left(\hat{n} - \vec{\beta}\right) a \cos \theta\,(1 - \cos \alpha) + a^2\,(1 - \beta \cos \alpha)^2$$
$$= a^2 \left[\cos^2 \theta \left(\beta^2 - 1\right) + (1 - \beta \cos \alpha)^2\right]\,.$$

Since the angular distribution of the power radiated is

$$\frac{dP_{\text{rad}}(\theta, \varphi, t_r)}{d\Omega} = \kappa \frac{dP_{\text{rad}}(\theta, \varphi, t)}{d\Omega} = \kappa |S_{\text{rad}}(\theta, \varphi, t)| R^2, \tag{3.58}$$

which implies that

$$\frac{dP_{\text{rad}}(\theta, \varphi, t_r)}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\left[\cos^2 \theta \left(\beta^2 - 1\right) + (1 - \beta \cos \alpha)^2\right]}{(1 - \beta \cos \alpha)^5}. \tag{3.59}$$

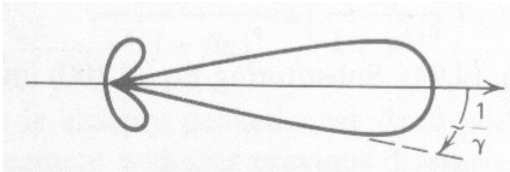

Figure 5: A cross section of the angular distribution of the power radiated when $\vec{a} \| \vec{v}$.

In spherical coordinates[9] recall that $\cos\alpha = \cos\varphi \sin\theta$. Putting this altogether and integrating the angular distribution gives

$$
\begin{aligned}
P_{\text{rad}}(t_r) &= \int \frac{dP_{\text{rad}}}{d\Omega} d\Omega & (3.60)\\
&= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_0^{2\pi} \left[\int_0^{\pi} \sin\theta \frac{\left[\cos^2\theta\,(\beta^2 - 1) + (1 - \beta\cos\alpha)^2\right]}{(1 - \beta\cos\alpha)^5} d\theta \right] d\varphi & (3.61)\\
&= \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{8\pi}{3} \frac{1}{(1-\beta)^2} & (3.62)\\
&= \frac{\mu_0 q^2 a^2}{6\pi c} \gamma^4. & (3.63)
\end{aligned}
$$

As above we see that the first part is the classical Larmor formula, with the relativistic correction $\gamma^4 = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^4}$. The integral identity

$$
\int_0^{2\pi} \left[\int_0^{\pi} \sin\theta \frac{\left[\cos^2\theta\,(\beta^2 - 1) + (1 - \beta\cos\alpha)^2\right]}{(1 - \beta\cos\alpha)^5} d\theta \right] d\varphi = \frac{8\pi}{3} \frac{1}{(1-\beta)^2} \tag{3.64}
$$

is slightly painful to establish and involves several substitutions. Checking it is left as an exercise to the motivated reader.

The emitted power,

$$
P_{\text{rad}}(t_r) = \frac{\mu_0 q^2 a^2}{6\pi c} \gamma^4, \tag{3.65}
$$

is known as the synchrotron radiation. In general if $\vec{a} \cdot \vec{v} = av\cos\vartheta$ the radiated power is

$$
P_{\text{rad}}(t_r) = \frac{\mu_0 q^2 a^2}{6\pi c} \left(\frac{1 - \beta\sin^2\vartheta}{(1-\beta^2)^3} \right), \tag{3.66}
$$

replicating the two earlier results for the in the limits $\vartheta = 0, \pi$.

3.4.3 Motion in a constant magnetic field

Synchrotron radiation is relevant when a charged particle is moving in a circle, such as a particle moving in a constant magnetic field.

[9]To match [2] instead use $\cos\theta = \cos\varphi \sin\alpha$ for spherical coordinates α, φ.

The relativistic equation of motion for an electron moving in a constant magnetic field is:

$$\gamma m_e \vec{a} = \frac{d}{dt}(\gamma m_e \vec{v}) = \frac{d\vec{p}}{dt} = -e\vec{v} \times \vec{B}, \tag{3.67}$$

so the acceleration is perpendicular to the velocity, e.g. the electron moves in a circle[10]. For uniform circular motion we have that the magnitude of the velocity is

$$v = \Omega_c \rho_L \tag{3.68}$$

with ρ_L the radius of the circle and Ω_c the angular velocity. We also have that the magnitude of the acceleration is

$$a = \Omega_c^2 \rho_L = \Omega_c v = \frac{evB}{\gamma m_e}, \tag{3.69}$$

and that the acceleration points inwards. This implies that the angular frequency is

$$\Omega_c = \frac{eB_0}{m_e \gamma} = \frac{\omega_c}{\gamma}, \tag{3.70}$$

where ω_c is the non-relativistic cyclotron frequency. We also find that the radius of the circular motion is

$$\rho_L = \frac{\gamma m_e v}{eB_0}, \tag{3.71}$$

known as the relativistic Larmor radius. The moving electron will emit synchotron frequency. A nice problem based on these ideas is to consider a classical version of the Bohr model of the atom, and estimates its life time using Equation (3.35). See Problem B.7.

3.5 Radiation Reaction

This section was not lectured but is included to show some of the subtleties that show up when particles are radiating. It is presented in more detail in [2].

In the previous sections we have observed that an accelerating particle radiates, this radiation carries away energy. Therefore, subject to the same force, a charged particle will accelerate less than a neutral particle of the same mass. This is because the emitted radiation exerts a force on the particle, $\vec{F}_{\text{rad}}$, known as the radiation reaction. We can attempt to derive the radiation reaction force from conservation of energy.

For a non-relativistic particle the Larmor formula, Equation (3.35) gives the total power radiated as

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}. \tag{3.72}$$

Conservation of energy then suggests that

$$\vec{F}_{\text{rad}} \cdot \vec{v} = -P = -\frac{\mu_0 q^2 a^2}{6\pi c}, \tag{3.73}$$

e.g. the particle loses energy due to the radiation reaction force. Unfortunately this equation is not complete. This is because we need to consider the total power lost, including power lost

[10]We have cheated slightly, γ is only constant if $|\vec{v}|$ is constant, e.g. the acceleration changes the direction of the velocity but not its magnitude. This is the case for a particle moving in a circle.

due to the velocity field. We can circumvent this by considering the time average of the power over an interval which has the same energy stored in the velocity field at both end points. The time averaged version of Equation (3.73),

$$\int_{t_1}^{t_2} \vec{F}_{\text{rad}} \cdot \vec{v} dt = -\frac{\mu_0 q^2}{6\pi c} \int_{t_1}^{t_2} a^2 dt \tag{3.74}$$

is then valid. To make more sense of the time average equation integrate by parts on the right hand side:

$$\int_{t_1}^{t_2} a^2 dt = \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt} dt = \left(\vec{v} \cdot \frac{d\vec{v}}{dt}\right)_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v} dt, \tag{3.75}$$

for the velocity field to store the same energy at both ends of the interval we are assuming that $\vec{v}(t_1) = \vec{v}(t_2)$, so the first term in the integration by parts vanishes and we are left with

$$\int_{t_1}^{t_2} \left(\vec{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}\right) \cdot \vec{v} dt = 0. \tag{3.76}$$

This leads to the Abraham-Lorentz formula

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}. \tag{3.77}$$

This equation does not include components of $\vec{F}_{\text{rad}}$ perpendicular to $\vec{v}$ so it is still not complete. However, it is good enough for a first attempt.

Equation (3.77) has some troubling implications:

1. If there are no external forces then Newton's second law implies that

$$F_{\text{rad}} = \frac{\mu_0 q^2 \dot{a}}{6\pi c} = ma, \tag{3.78}$$

   which is solved by $a = a_0 e^{\frac{t}{\tau}}$, with $\tau = \frac{\mu_0 q^2}{6\pi c}$. For an electron $\tau = 6 \times 10^{-24} s$. This means that the acceleration spontaneously increases leading to runaway solutions.

2. The above runaway solutions can be excluded if $a_0 = 0$. However, this has the unpleasant consequence that if an external force is applied, the charged particle accelerates before feeling the force.

These are both pathological and undesirable features, suggesting that something is wrong with our classical theory of a charged particle.

There is relativistic generalisation of Equation (3.77), however, it does not solve the problems. Theses pathologies suggest that there is no such thing as a point charge in classical electromagnetism.

For a discussion of the physical mechanism behind the radiation reaction the interested reader is referred to [2].

4 Different Types of Radiation

4.1 Cyclotron and Synchrotron

4.1.1 Qualitative overview

Before getting in to the details we can give a brief overview of cyclotron and synchrotron radiation.

Non-relativistic motion (cyclotron radiation). Consider a charge moving in a circle, e.g. in a constant magnetic field. Consider the observer being in the plane of rotation, as the radiation field is proportional to $\hat{n} \times (\hat{n} \times \vec{a})$ it projects out the pieces of $\vec{a}$ that are parallel to $\vec{v}$ and we are left with the perpendicular piece of $\vec{a}$.

Include a figure here!

Observers will see the rotating charge as being an oscillating dipole with dipole moment

$$d(t) = d_0 \sin(\omega_c t + \phi_0) = q\rho_L \sin(\omega_c t + \phi_0) . \tag{4.1}$$

This implies that the radiation field is

$$\begin{aligned} \vec{E}_{\rm rad} &= \frac{\mu_0 q}{4\pi} \frac{\hat{n} \times (\hat{n} \times \vec{a})}{r} && (4.2)\\ &= \frac{\mu_0 q}{4\pi} \frac{|a| \sin(\omega_c t + \phi_0)}{r} \hat{e} && (4.3)\\ &= \frac{m_0 q}{4\pi} \rho_L \omega_c^2 \frac{(\omega_c t + \phi_0)}{r} \hat{e}. && (4.4) \end{aligned}$$

This is purely a sinusoidal oscillation and thus its Fourier transform consists of a single delta function peak. To see this consider that

$$\sin \omega_c t = \frac{1}{2i} \left(e^{i\omega_c t} - e^{-i\omega_c t} \right) , \tag{4.5}$$

and the Fourier transform of the exponential is

$$\mathrm{FT} \left(e^{i\omega_c t} \right) = 2\pi \delta (\omega - \omega_c) , \tag{4.6}$$

a Dirac delta function. Thus the Fourier transform of $\vec{E}_{\rm rad}$ consists of the sum of two delta functions $\delta(\omega - \omega_c) + \delta(\omega + \omega_c)$, and for positive frequency there is a single peak at $\omega = \omega_c$. The power spectrum is proportional to the square of the radiation field, $P(\omega) \sim |\mathrm{FT}[E(t)](\omega)|^2$, thus also consists of a single peak at ω_c.

The takeaway from this example is that the cyclotron frequency directly determines the frequency of emitted radiation.

Relativistic motion (synchrotron radiation). We know that for a relativistic particle with acceleration perpendicular to velocity that the radiation is sharply peaked along $\vec{v}$. This is sometimes called lighthouse type emission, or headlight emission. It is similar to the above case in that the relativistic cyclotron frequency Ω_c determines the spectrum. However, the relativistic correction factors mean that now there is radiation at every harmonic of Ω_c, $\omega = n\Omega_c$ for $n \in \mathbb{Z}$. The dominant contribution is not at $\omega = \Omega_c$ but at a higher frequency, e.g. $\omega = 7\Omega_c$. The higher harmonics of Ω_c correspond to extremely large photon energies for ultra relativistic particle, $\gamma \gg 1$.

4.1.2 Basic mathematical details

Consider a relativistic massive particle, with mass m and charge q, moving in a constant magnetic field. The equation of motion is

$$\frac{d}{dt}\left(\gamma m\vec{v}\right) = q\vec{v} \times \vec{B}, \tag{4.7}$$

this is a relativistic analogue of the Lorentz force law for a magnetic force. There is a corresponding "time component" of this equation

$$\frac{d}{dt}\left(\gamma mc^2\right) = q\vec{v} \cdot \vec{E} = 0. \tag{4.8}$$

Another way to phrase this second equation is that there is no work done by a magnetic force. This implies that the magnitude of the velocity, and thus the gamma factor, is constant. Importantly, the velocity is not constant, it is only the direction which is able to change. Rearranging the equation of motion and solving for the acceleration we find

$$\vec{a} = \frac{q}{\gamma m}\vec{v} \times \vec{B}, \tag{4.9}$$

with magnitude

$$a = \frac{qvB}{\gamma m}\sin\alpha, \tag{4.10}$$

where α is the angle between $\vec{v}$ and $\vec{B}$ called the pitch angle. Working in the rest frame of the charge, we know that the emitted power is given by the Larmor formula

$$P' = \frac{q^2\left(a'\right)^2}{6\pi\varepsilon_0 c^3} = \frac{\mu_0 q^2\left(a'\right)^2}{6\pi c}, \tag{4.11}$$

using $c^2 = \frac{1}{\varepsilon_0\mu_0}$ in the last equality, with a' the acceleration in this frame. A natural question is how this is related to the power emitted in an observer's frame[11]. Recall that energy and time transform the same way when moving between rest frames;

$$dE = \gamma dE', \tag{4.12}$$

$$dt = \gamma dt'. \tag{4.13}$$

Thus

$$\frac{dE'}{dt'} = \frac{dE}{dt}, \qquad \Rightarrow P' = P, \tag{4.14}$$

[11] In principle this is a straight forward exercise in using the relativistically corrected power emitted formula in Eq. (3.65). However, here we give a bit more explanation.

so the same power is emitted. Finally we need to use the relationship between the acceleration in the two frames to express the power radiated in terms of the a that we found above. Recall that

$$a'_{\parallel} = \gamma^3 a_{\parallel}, \quad \text{parallel to motion} \tag{4.15}$$

$$a'_{\perp} = \gamma^2 a_{\perp}, \quad \text{Perpendicular to motion.} \tag{4.16}$$

Here the acceleration is perpendicular to the velocity so $a'_{\parallel} = 0$ and $a' = \gamma^2 a$. Putting all of this together the power radiated in the observers frame is

$$P = \frac{q^2\gamma^4 a^2}{6\pi\varepsilon_0 c^3} = \frac{q^4\gamma^2 v^2 B^2 \sin^2\alpha}{6\pi\varepsilon_0 m^2 c^3}. \tag{4.17}$$

This is the power radiated by a single charge moving with pitch angle α. For a collection of charges with isotropic velocities we need to average over the pitch angle. This is done by integrating over the solid angle $d\Omega = r\sin\alpha d\alpha d\varphi$ and normalising by the volume of the radius r S^2, $4\pi r$,

$$\langle \sin^2\alpha \rangle = \frac{1}{4\pi r}\int \sin^2\alpha d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\alpha d\alpha = \frac{2}{3}. \tag{4.18}$$

Which implies that

$$\langle P \rangle = \frac{q^4\gamma^2 v^2 B^2}{9\pi\varepsilon_0 m^2 c^3}. \tag{4.19}$$

This is inversely proportional to the mass squared, $\langle P \rangle \sim m^{-2}$. Thus the lighter the charged particle the more power it radiates. Because of this synchrotron radiation from electrons is often the dominant contribution, over the radiation due to protons or ions. This is particularly relevant in particle accelerators, and is the reason why protons in the LHC achieve much higher energies than electron in did in LEP even though they use the same tunnel.

4.1.3 Spectrum and general properties

The angular distribution of the synchrotron radiation for each particle is highly doppler shifted in the forward direction, as can be seen in Figure. 5. The radiation from a single charge appears as a pulse with duration, Δt, much less than the gyration period. This is because there is only a small arc of the charges circular orbit where the observer's line of sight lies inside the emission cone.

Recall that the Fourier transform relates a narrow function of time to a broad function of frequency, as $\Delta t \Delta\omega = 4\pi$. Thus we expect that there is a broad frequency spectrum. To get a rough estimate of Δt using Figure. 6

Recall that the equation of motion is

$$m\gamma\frac{dv}{dt} = qvB\sin\alpha, \tag{4.20}$$

or alternatively

$$ma = \frac{qvB\sin\alpha}{\gamma} = \frac{mv^2}{R}, \tag{4.21}$$

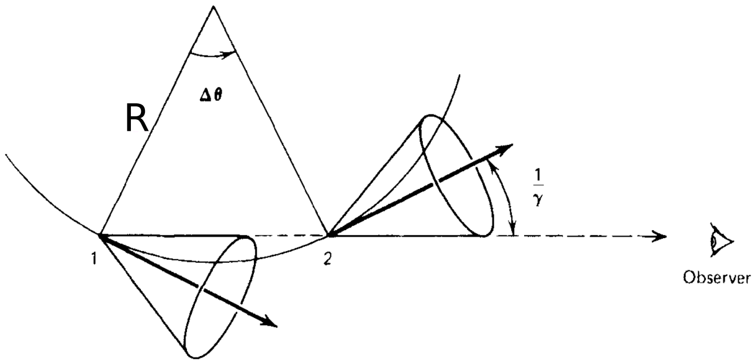

Figure 6: Consider a small arc of the charge's circular orbit. Point 1 is where the emission begins to be visible to the observer and point 2 is where the emission stops being visible. This means that our small angle is given by $\Delta\theta = \frac{2}{\gamma}$, and the approximation $d \simeq \Delta s$ is valid for ultra relativistic charges, $\gamma \gg 1$. For the time interval $\Delta t = t_2 - t_1$ the observers' line of sight lies inside the emission cone. Figure taken from [1].

where the final equality is to the centipetal force for a circular orbit of radius R. Solving for R gives

$$R = \frac{m\gamma v}{qB\sin\alpha} = \frac{v}{\omega_B \sin\alpha}, \tag{4.22}$$

where $\omega_B = \frac{qB}{\gamma m}$ is the gyration frequency of the charge's orbit. For the distance the charge travels along the arc between point 1 and point 2 consider the definition of an angle in radians,

$$\Delta s = R\Delta\theta = \frac{2v}{\gamma\omega_B \sin\alpha}. \tag{4.23}$$

In the ultra relativistic limit, we also have that

$$\Delta s \sim v\Delta t, \tag{4.24}$$

which implies that

$$v\Delta t \simeq \frac{2v}{\gamma\omega_B \sin\alpha}. \tag{4.25}$$

This gives the pulse duration as

$$\Delta t = \frac{2}{\gamma\omega_B \sin\alpha}. \tag{4.26}$$

This is not the same as the difference in arrival times measured by an observer between a photon (p_1) emitted at point 1 and a photon (p_2) emitted at point 2. This difference is determined by the difference between the speed of light and the speed of the charge. The difference in arrival times is Δt^A, it is related to the difference in emission times, Δt, through

$$c\Delta t^A = c\Delta t - \Delta s. \tag{4.27}$$

Which gives that

$$\Delta t^A = \frac{2}{\gamma\omega_B \sin\alpha}\left(1 - \frac{v}{c}\right). \tag{4.28}$$

This can be rewritten in terms of the γ factor in the following way:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}, \tag{4.29}$$

$$\Rightarrow \frac{v}{c} = \left(-\frac{1}{\gamma^2}\right)^{\frac{1}{2}} \tag{4.30}$$

$$\simeq 1 - \frac{1}{2\gamma^2}, \quad \text{for } \gamma \gg 1. \tag{4.31}$$

Thus we have that $1 - \frac{v}{c} \simeq \frac{1}{2\gamma^2}$ and

$$\Delta t^A = \frac{1}{\gamma^3 \omega_B \sin\alpha}. \tag{4.32}$$

On the other hand the period of gyration is given by

$$T = \frac{2\pi}{\omega_B}, \tag{4.33}$$

so that

$$\Delta t^A = \frac{1}{\gamma^3} T \frac{1}{2\pi \sin\alpha}. \tag{4.34}$$

The pulse is thus very narrow for $\gamma \gg 1$. As a consequence of this, we expect the spectrum to have an effective cut off at an angular frequency of order $\frac{1}{\Delta t^A}$, known as the critical frequency. Above this cut off the spectrum will be negligible. Conventionally the cut off (angular) frequency is taken to be

$$\omega_c = \frac{3}{2}\gamma^3 \omega_B \sin\alpha, \qquad \text{or } \nu_c = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha. \tag{4.35}$$

Previously we considered the total power, what about the qualitative behaviour of the spectrum?

Introduce a dimensionless function $F\left(\frac{\omega}{\omega_c}\right)$ so that that the power spectrum can be written as

$$P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right), \tag{4.36}$$

with C_1 a constant term containing the various physical constants and specified parameters in the power. This is the Fourier transform of the power radiated by a charge with tilt angle α. The total power is the integral over the spectrum,

$$P = \int_0^\infty P(\omega) d\omega = C_1 \int_0^\infty F\left(\frac{\omega}{\omega_c}\right) d\omega. \tag{4.37}$$

If we can turn this integral into a definite integral, it will be a constant piece and we can find the expression for the constant. To do this let $x = \frac{\omega}{\omega_c}$ then $d\omega = \omega_c dx$ which implies

$$P = C_1 \omega_c \int_0^\infty F(x) dx, \tag{4.38}$$

giving us our desired definite integral. For a method of determining the function $F(x)$ see Chapter 6 of [1]. The constant piece is thus

$$
\begin{aligned}
C_1 &= \frac{P}{\omega_c}\left[\int_0^\infty F(x)dx\right]^{-1} \\
&= \left(\frac{q^4\gamma^2 v^2 B^2 \sin^2\alpha}{6\pi\varepsilon_0 m^2 c^3}\right)\frac{1}{\frac{3}{2}\gamma^3\omega_B\sin\alpha}\left[\int_0^\infty F(x)dx\right]^{-1} \\
&= \frac{q^3 B\sin\alpha}{9\pi\varepsilon_0 mc}\left[\int_0^\infty F(x)dx\right]^{-1},
\end{aligned}
\tag{4.39}
$$

note that the factors of γ have cancelled out, we have approximated $v^2 \simeq c^2$, and substituted in the relativistic cyclotron frequency $\omega_B = \frac{qB}{\gamma m}$.

The power spectrum is thus

$$
P(\omega) = C_n \frac{q^3 B\sin\alpha}{9\pi\varepsilon_0 mc} F\left(\frac{\omega}{\omega_c}\right), \tag{4.40}
$$

here C_n is the normalisation constant determined by F. The only γ dependence is in the ω_c.

Example 4.1. Power Law Spectrum Often the observed spectrum has a power law behaviour over a certain frequency range:

$$
P(\omega) \propto \omega^{-s}, \qquad \omega \in (\omega_1, \omega_2). \tag{4.41}
$$

The exponent s is known as the spectral index.

This happens if the distribution of particle energies, or equivalently the distribution of Lorentz factors, is power law:

$$
N(E)dE = CE^{-p}dE, \qquad E \in (E_1, E_2) \tag{4.42}
$$

$$
N(\gamma)d\gamma = \tilde{C}\gamma^{-p}d\gamma, \qquad \gamma \in (\gamma_1, \gamma_2). \tag{4.43}
$$

The quantities $C, \tilde{C}$ are not necessarily constant as they can depend on the pitch angle α.

For a collection of charges the total power is

$$
P_{\text{tot}}(\omega) = \int_{\gamma_1}^{\gamma_2} P_1(\omega)N(\gamma)d\gamma, \tag{4.44}
$$

with $P_1(\omega)$ the power spectrum of a single charge, which we computed above. Substituting in our expression for P_1 gives

$$
P_{\text{tot}}(\omega) = C\int_{\gamma_1}^{\gamma_2} P_1(\omega)\gamma^{-p}d\gamma \propto \int_{\gamma_1}^{\gamma_2} F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma. \tag{4.45}
$$

Evaluate this using the change of variables $x = \frac{\omega}{\omega_c} = \kappa\omega\gamma^{-2}$ with κ a constant. In terms of γ this means that $\gamma = \sqrt{\frac{\kappa\omega}{x}}$ which implies that $d\gamma = -\frac{1}{2}\sqrt{\kappa\omega}x^{-\frac{3}{2}}dx$ and

$$
P_{\text{tot}}(\omega) \propto \int_{x_1}^{x_2} F(x)\left(\omega^{-\frac{p}{2}}x^{\frac{p}{2}}\right)\left(\omega^{\frac{1}{2}}x^{-\frac{3}{2}}\right)dx \propto \omega^{-\frac{p-1}{2}}\int_{x_1}^{x_2} F(x)x^{\frac{p-3}{2}}dx. \tag{4.46}
$$

The limits of integration correspond to the limits of the region where the distribution is a power law. These limits can depend on the frequency, thus the integral is not necessarily a constant. However, if the interval (γ_1, γ_2) is wide enough then we can approximate $x_1 \simeq 0$ and $x_2 \simeq \infty$. This means that the definite integral is a constant and the power spectrum is

$$P_{\text{tot}} \propto \omega^{-\frac{p-1}{2}}. \tag{4.47}$$

This is a power law spectrum with spectral index $s = \frac{p-1}{2}$. When we see an example of the power spectrum later we will see that there is a power law regime for higher frequencies. This regime is often called optically thin as the distribution of charges is transparent to photons, they pass easily through the charges.

Lifetime of Synchrotron Sources: Consider a distribution of charges emitting synchrotron radiation. The individual charges lose energy through radiation. This means that they can only radiate for a finite amount of time before they loose all of their energy. An estimate of the lifetime of a radiating charge is

$$\tau_{\text{synch}} \simeq \frac{E}{\langle P \rangle} = \frac{\gamma m c^2}{\frac{q^4 \gamma^2 v^2 B^2}{9\pi\varepsilon_0 m^2 c^3}} = \frac{9\pi\varepsilon_0 m^3 c^5}{e^4 \gamma v^2 B^2}. \tag{4.48}$$

Looking at this we see that the more energetic a charge is the more powerful its radiation is and the shorter its lifetime is.

In Figure 7, taken from [1], a plot of the dimensionless function F is given. This gives the shape of the power spectrum, and we see the peak at $\omega_{\text{peak}} = 0.3\omega_c \simeq \frac{1}{2}\frac{eB}{m}\gamma^2 \sin\alpha$.

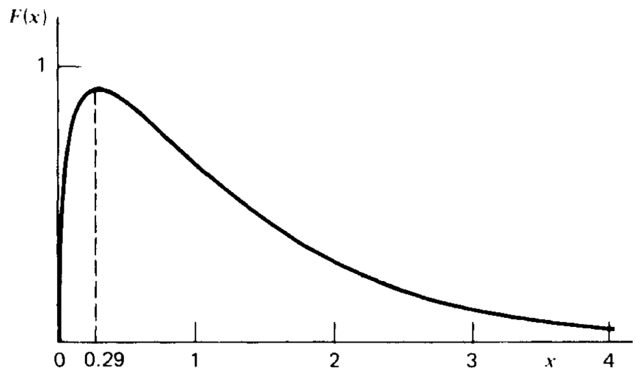

Figure 7: A plot of the dimensionless function F from [1].Here $x = \frac{\omega}{\omega_c}$. The peak is at $\omega = 0.29\omega_c$.

Thus more energetic charges radiate at higher frequencies. Because the charges radiating at higher frequencies have shorter lifetimes the spectrum steepens with time.

An example of a spectrum with a power law regime is in Figure 8, again the figure is taken from [1]. In the optically thick regime the synchrotron radiation is readily reabsorbed by the population of charges that emitted it. This is known as synchrotron self-absorption.

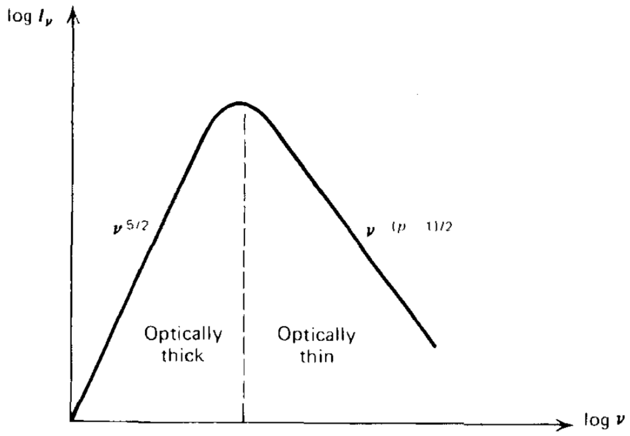

Figure 8: A plot of intensity, power spectrum, for a group of charges with a power law distribution of energy. The two regimes are: in the optically thick regime charges are opaque to radiation as it is readily reabsorbed, in the optically thin regime the charges are transparent to the radiation and it is emitted with a power law spectrum.

Polarisation of Synchrotron Radiation. Charges in the background of a magnetic field gyrate, rotate perpendicular to the direction of the local magnetic field. This determines the direction of linear polarisation for the synchrotron radiation in the optically thin regime.

Averaged over the tilt angle, the left and right handed elliptically polarised components of the radiation cancel out. This leaves linearly, or at least partially linearly, polarised radiation. The radiation of two components:

1. $\chi \perp B$, the projection of B onto the plane of the sky is orthogonal to the direction of linear polarisation.
2. $\chi \| B$, the direction of linear polarisation is parallel to the plane of the sky.

This is shown in Figure 9

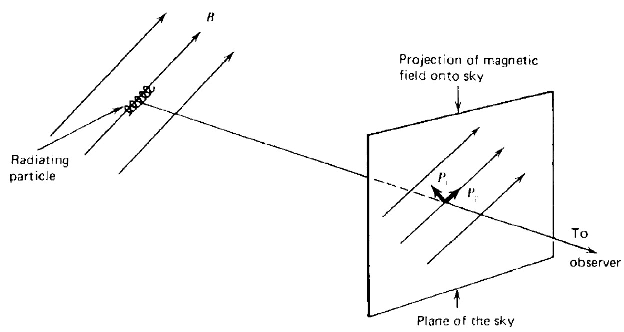

Figure 9: Taken from [1], a sketch of the decomposition of the polarisation of synchrotron radiation into components perpendicular to and parallel to the plane of the sky.

It turns out that option 1 is more efficient at both emission and absorption. This means that in the optically thin regime $\chi \perp B$ is preferentially emitted, so is the dominant contribution to the radiation. This has a high degree of polarisation, $\Pi(\omega)$ can be as high as 75%.

In the optically thick regime, $\chi \perp B$ is preferentially absorbed. This leaves $\chi \| B$ as the dominant contribution to the radiation. The degree of polarisation is now around 10% to 15%.

Summary/ Hallmarks of Synchrotron Radiation.

- The dominant contribution is from lighter charges particles, such as electrons, since $\langle P\rangle \sim \frac{1}{m^2}$.
- The radiation from a single particle lies within $\sim \frac{1}{\gamma}$ of a cone with half angle the pitch angle α.
- When the number density of particles with energy E has a power law dependence, $N(E) \sim E^{-p}$ (or equivalently $N(\gamma) \sim \gamma^{-p}$), the spectrum has a power law behaviour
$$P_{\text{tot}} \sim \omega^{-s}, \tag{4.49}$$
for $s = \frac{p-1}{2}$ the spectral index.
- Synchrotron radiation has a characteristic linear polarisation of up to 70% in the optically thin region, where the spectrum is a power law.

4.2 Bremsstrahlung

The radiation emitted when a charge is deflected or decelerated by other charges, e.g. an electron in the presence of ions, is known as Bremsstrahlung. There are two simplified cases

(A) Deceleration: An electron comes to a stop near an ion of charge Ze.

(B) Deflection: An electron is deflected when it passes near an ion. The direction of deflection depends on the charge of the ion.

Include figures.

In both cases the interaction takes place over a finite time interval. This leads to characteristic spectrum of radiation for Bremsstrahlung. From our earlier analysis, we know that the emitted radiation is proportional to the acceleration of a non-relativistic charge[12]. The radiation field in this case is
$$\vec{E}_{\text{rad}} = \frac{\mu_0 q}{4\pi}\frac{[\hat{n}\times(\hat{n}\times\vec{a})]}{r}. \tag{4.50}$$
Treating the radiating electron as an oscillating dipole, we know that the power emitted by radiation is
$$P_{\text{rad}}(\omega) = \frac{\mu_0}{6\pi c}\omega^4\left|\vec{d}(\omega)\right|^2, \tag{4.51}$$

[12]This is also true for a relativistic charge where we also need to include the γ factor correction.

with ω the oscillation frequency. As it is a function of frequency, we refer to $P_{\text{rad}}(\omega)$ as the spectrum. Interpret $\vec{d}(\omega)$ as the Fourier transform of the dipole moment, $\vec{d}(t) = q\vec{r}(t)$. Computing this Fourier transform we have that,

$$
\begin{aligned}
\left|\vec{d}(\omega)\right|^2 &= \left|\int_{-\infty}^{\infty} \vec{d}(t)e^{i\omega t}dt\right|^2 && (4.52)\\
&= \left|\frac{1}{i\omega}\left[\int_{-\infty}^{\infty}\frac{d}{dt}\left(\vec{d}(t)e^{i\omega t}\right)dt - \int_{-\infty}^{\infty}\dot{\vec{d}}(t)e^{i\omega t}dt\right]\right|^2, \quad \text{after integrating by parts} && (4.53)\\
&= \frac{1}{\omega^2}\left|\int_{-\infty}^{\infty}\dot{\vec{d}}(t)e^{i\omega t}dt\right|^2 && (4.54)\\
&= \frac{1}{\omega^4}\left|\int_{-\infty}^{\infty}\left[\frac{d}{dt}\left(\dot{\vec{d}}e^{i\omega t}\right) - \ddot{\vec{d}}(t)e^{i\omega t}\right]dt\right|^2, \quad \text{again integrate by parts} && (4.55)\\
&= \frac{1}{\omega^4}\left|\text{FT}\left[\ddot{\vec{d}}(t)\right](\omega)\right|^2. && (4.56)
\end{aligned}
$$

Thus

$$\left|\vec{d}(\omega)\right|^2 = \frac{q^2}{\omega^4}\left|\text{FT}\left[\vec{a}(t)\right](\omega)\right|^2. \tag{4.57}$$

In the integration by parts the boundary terms vanish as we assume that the dipole moment and its derivative vanish asymptotically. This calculation tells us that to find the radiation spectrum we need to understand the Fourier transform of the acceleration.

To carry this out we focus on case (B) in the weak deflection limit, e.g. v_x is large enough that the charge is not deflected very much and we can assume that v_x is constant. For simplicity we will also take the ion to be negatively charged when we draw the figure.

Include a figure here!

This is a scattering problem with the deflection depending on the impact parameter, sometimes called the scattering parameter, b. The impact parameter is the distance between the moving charge and the ion perpendicular to the charge's initial velocity.

Consider the velocity before and after deflection:

- Before:

$$\vec{v} = \begin{pmatrix} v_x \\ 0 \end{pmatrix}. \tag{4.58}$$

- After:

$$\vec{v}' = \begin{pmatrix} v'_x \\ \Delta v_y \end{pmatrix}. \tag{4.59}$$

In the weak deflection limit we assume that $v_x(t) = v_x$ is a constant, e.g. the horizontal velocity does not change, and that $\Delta v_y(t) \ll v_x$, the vertical velocity and hence the deflection is negligible. We can thus take the position vector to be

$$\vec{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \simeq \begin{pmatrix} v_x t \\ b \end{pmatrix}. \tag{4.60}$$

Inserting this into the equation of motion, and projecting onto the vertical direction gives:

$$ma_y(t) = \vec{F}_{\text{Coulomb}} \cdot \hat{e}_y \tag{4.61}$$
$$= -\frac{Zq^2}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3} \cdot \hat{e}_y \tag{4.62}$$
$$= -\frac{Zq^2}{4\pi\varepsilon_0}\frac{b}{\left[(v_x t)^2 + b^2\right]^{\frac{3}{2}}}. \tag{4.63}$$

From this we observe that $a_y(t)$ is peaked around $t = 0$ and has a width of $\Delta t = \frac{b}{v_x}$. This is already enough to tell us that the Fourier transform will also have a peak with the inverse width $\Delta\omega \simeq 4\pi\frac{v_x}{b}$. Thus is $a_y(t)$ is sharply peaked $a_y(\omega)$ will be broad. It turn out the value of $a_y(\omega)$ is related to Δv_y, which is found by integrating $a_y(t)$.

Carrying out this integration we have,

$$\Delta v_y = \int_{-\infty}^{\infty} a_y dt \tag{4.64}$$
$$= -\frac{Zq^2}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{b}{\left[(v_x t)^2 + b^2\right]^{\frac{3}{2}}}dt \tag{4.65}$$
$$= -\frac{Zq^2}{4\pi\varepsilon_0}\frac{1}{b^2}\int_{-\infty}^{\infty}\frac{1}{\left[\left(\frac{v_x}{b}t\right)^2 + 1\right]^{\frac{3}{2}}}\frac{b}{v_x}d\left(\frac{v_x}{b}t\right) \tag{4.66}$$
$$= -\frac{Zq^2}{4\pi\varepsilon_0}\frac{1}{v_x b}\int_{-\infty}^{\infty}\frac{1}{(u^2+1)^{\frac{3}{2}}}du \tag{4.67}$$
$$= -\frac{Zq^2}{2\pi\varepsilon_0}\frac{1}{v_x b} \tag{4.68}$$

where we have used the standard integral $\int_{-\infty}^{\infty}\frac{1}{(u^2+1)^{\frac{3}{2}}}du = 2$.

To understand the qualitative shape of the Bremsstrahlung spectrum we can approximate $a_y(\omega)$ by a box function[13]. That is we take $a_y(\omega)$ to be a step function that is only non-zero for $|\omega| < \frac{2\pi v_x}{b}$. The step height is taken to be maximum value, $|a_y(\omega = 0)|$:

$$|a_y(\omega = 0)| = \left|\int_{-\infty}^{\infty} a_y(t)e^{i(0)t}dt\right| = \left|\int_{-\infty}^{\infty} a_y(t)dt\right| = |\Delta v_y|\,, \tag{4.69}$$

exactly what we calculated above. The approximation of the spectral shape is thus

$$|a_y(\omega)| \simeq \begin{cases} |\Delta v_y|^2 & \text{if } |\omega| < \dfrac{2\pi v_x}{b} \\ 0 & \text{else,} \end{cases} \tag{4.70}$$

with

$$|\Delta v_y|^2 = \frac{Z^2 q^4}{4\pi^2\varepsilon_0^2 m^2 v_x^2}\frac{1}{b^2}. \tag{4.71}$$

[13]There are other approximates that can be used. However, I will claim that they all replicate the same qualitative features. The box function is the simplest.

For different impact parameters the spectral shape will change. However, the general features remain the same, the spectra are relatively flat and broad up to a cut off frequency set by v_x and b. In reality the spectrum is an ensemble of scattering events for different impact parameters, there is also a low frequency cut off due to absorption of emitted photons.

5 Plasma Physics

So far we have mostly considered radiation in vacuum. Often the presence of free charges is important in astrophysics. Thus we need to consider a globally neutral ionised gas, a plasma. Plasmas occur in many circumstances; lightning, solar corona, fluorescent lights, fusion reactors, etc.

A plasma is made up of two, effectively, non-interacting gases . The negatively charged electrons, and the positively charged ions. Usually the ions are so much more massive that we can approximate them as being stationary.

5.1 Electromagnetic Waves in a Plasma

In a plasma there is a natural oscillation frequency associated with the movement of the electrons.

Start from the quasi-neutral plasma, and consider a slab of plasma. Imagine displacing all of the electrons in the slab by a small amount x, assuming that the ions stay in place. This results in two charged slabs: the negatively charged slab with the displaced electrons, and the positively charged slab where the electrons used to be. This results in an effective capacitor with the charge on each plate being

$$Q = \pm e n_e A x, \tag{5.1}$$

with e the charge of an electron, n_e the concentration of electrons in the plasma, and A the cross sectional area of the slabs, The charge density of the slab is

$$\sigma_q = \pm e n_e x. \tag{5.2}$$

Figure of slab to be added.

The induced electric field between the slabs is

$$\vec{E} = \frac{\sigma_q}{\varepsilon_0}\hat{x} = \frac{e n_e x}{\varepsilon_0}\hat{x}. \tag{5.3}$$

Writing down the equation of motion for an electron we have:

$$m_e a = m_e \frac{d^2 x}{dt^2} = eE = -\frac{e^2 n_e x}{\varepsilon_0}. \tag{5.4}$$

This means that the electrons feel a restoring force back towards their initial position. The electrons thus undergo simple harmonic motion with frequency

$$\omega^2 = \frac{e^2 n_e}{m_e \varepsilon_0} = \omega_{pe}^2. \tag{5.5}$$

This is known as the plasma frequency[14].

Now consider an electromagnetic wave travelling through a plasma with electron concentration (number density) n_e. The initial electric field is

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}. \tag{5.6}$$

The force due to the magnetic component is much smaller than that due to the electric field so we discount it here. The equation of motion for an electron in the plasma is thus

$$\begin{aligned} m_e \frac{d\vec{v}}{dt} &= -e\vec{E} \\ &= -e\vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} \\ \Rightarrow \vec{v} &= \frac{e}{im_e\omega}\vec{E}, \end{aligned}$$

where we have integrated over time to find $\vec{v}$. It is an oscillating quantity with the same frequency as the external electromagnetic wave.

The current density for electrons in the plasma due to the external field is

$$\vec{j} = -n_e\vec{v} = -\frac{n_e e^2}{i\omega m_e} = \sigma\vec{E}, \tag{5.7}$$

with σ the conductivity. The conductivity of the plasma is thus

$$\sigma = \frac{in_e e^2}{\omega m_e}. \tag{5.8}$$

As a plasma is a conductive medium the wave number is complex:

$$\begin{aligned} k^2 &= \frac{\omega^2}{c^2} + i\mu_0\sigma\omega \\ &= \frac{\omega^2}{c^2} + i\mu_0\omega\left(\frac{in_e e^2}{m_e\omega}\right) \\ &= \frac{\omega^2}{c^2} - \mu_0\varepsilon_0\omega_{pe}^2 \\ &= \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right). \end{aligned}$$

When $\omega_{pe} > \omega$ $k^2 < 1$, and the electromagnetic wave is exponentially decaying. This is referred to as the no propagation regime.

When $\omega > \omega_{pe}$ we have $k^2 > 1$ and there is propagation of the electromagnetic wave. e.g. an electromagnetic wave only propagates through a plasma if its frequency is larger than the plasma frequency.

[14]It is the electron plasma frequency. There is also a plasma frequency for the ions, $\omega_{pi}^2 = \frac{Z^2e^2n_i}{m_i\varepsilon_0}$. This is derived in the same way as the electron plasma frequency but now we need to be careful about the direction of the electric field.

For a propagating electromagnetic wave the phase velocity is

$$v_{\rm ph} = \frac{\omega}{c} = c\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{-\frac{1}{2}} = \frac{c}{n_r}, \tag{5.9}$$

where the refractive index is defined as

$$n_r = \sqrt{\varepsilon} = \sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}, \tag{5.10}$$

in terms of the permittivity ε.

A propagating wave has $\omega > \omega_{pe}$ and thus $n_r < 1$. This implies that $v_{\rm ph} > c$, which seems to violate special relativity!

Fortunately information does not travel at the phase velocity, it travels at the group velocity[15] The group velocity is defined as

$$v_g = \frac{\partial\omega}{\partial k} = c\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}} = cn_r. \tag{5.11}$$

As $\omega > \omega_{pe}$ the group velocity is smaller than c.

5.2 Dispersion Measure

From Equation (5.11) the group velocity in the plasma depends on the frequency of the electromagnetic wave. This means that a pulse of radiation (a collection of electromagentic waves with different frequencies) that is initially narrow in ω spreads out (disperses) as it travels through a plasma. As an example of this consider a pulse emitted by a Pulsar that passed through the interstellar plasma on its way to earth.

Example 5.1. Pulsar Pulses
The time required for a signal, with frequency ω, to travel from its source to earth depends on the frequency. This is because the distance travelled is related to the group velocity through

$$v_g = \frac{ds}{dt}, \tag{5.12}$$

which implies that

$$t_\omega = \int_0^d \frac{ds}{v_g}, \tag{5.13}$$

with d the distance between the source and Earth. As the wave propagates through the interstellar plasma we can approximate $\omega \gg \omega_{pe}$. In other words the electron plasma frequency is much lower than the frequency of the signal.

This approximation allows us to expand the group velocity,

$$\frac{1}{v_g} = \frac{1}{c}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{-\frac{1}{2}} \simeq \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\right). \tag{5.14}$$

[15] The phase velocity is the velocity that the phase of the wave moves at, while the group velocity is the velocity of the wave packet.

This implies that the travel time is

$$t_\omega = \frac{1}{c}\int_0^d ds + \frac{1}{2c}\int_0^d \frac{\omega_{pe}^2}{\omega^2} ds \tag{5.15}$$

$$= \frac{d}{c} + \frac{e^2}{2cm_e\varepsilon_0}\frac{1}{\omega^2}\int_0^d n_e ds, \quad \text{substituting in } \omega_{pe}^2. \tag{5.16}$$

The first term is the travel time in vacuum, and the second term is the correction due to the presence of the plasma. The electron concentration is left in the integral because it can vary along the path. The integral of the electron concentration over the path travelled is called the dispersion measure,

$$\mathcal{D} = \int_0^d n_e ds. \tag{5.17}$$

Typically it is the rate of change of arrival time with frequency that is measured:

$$\frac{t_\omega}{d\omega} = -\frac{e^2}{cm_e\varepsilon_0\omega^3}\mathcal{D}. \tag{5.18}$$

Thus $\mathcal{D}$ is computed from the measured $\frac{t_\omega}{d\omega}$. If we assume a typical value of n_e along the path we can estimate the distance from the source to the Earth,

$$\mathcal{D} = \int_0^d n_e \simeq (n_e)_{\text{typ}}\, d, \qquad \Rightarrow d \simeq \frac{\mathcal{D}}{(n_e)_{\text{typ}}}. \tag{5.19}$$

This means that if we can measure d through a method like parallax, then we can estimate n_e along the line of sight.

Considering examples from the *ATNF Pulsar catalogue*, we can infer that the typical electron concentration is

$$(n_e)_{\text{typ}} \simeq 0.02 - 0.04\text{cm}^{-3}. \tag{5.20}$$

We can also infer that, in the absence of evidence that the line of sight passes through regions of high electron concentration, a high dispersion measure implies that the source is far away.

5.3 Faraday Rotation

What happens if the plasma is subject to a fixed external magnetic field? The properties of an electromagnetic wave propagating through the plasma now depends on its direction of travel relative to the direction of the magnetic field.

Think of the wave as a superposition of a left circularly polarised and a right circularly polarised component:

$$\vec{E}_L(t) = E_0\left[\cos\omega t\,\hat{\epsilon}_1 + \sin\omega t\,\hat{\epsilon}_2\right], \tag{5.21}$$

$$\vec{E}_R(t) = E_0\left[\cos\omega t\,\hat{\epsilon}_1 - \sin\omega t\,\hat{\epsilon}_2\right], \tag{5.22}$$

here $\hat{\epsilon}_1, \hat{\epsilon}_2$ are the polarisation unit vectors perpendicular to the propagation direction.

A nice compact expression that stands for both components[16] is

$$\vec{E}(t) = E_0 e^{-i\omega t} \left(\hat{\epsilon}_1 \mp \hat{\epsilon}_2\right), \tag{5.23}$$

with + for RCP and − for LCP.

If we assume that the local magnetic field is much stronger than the magnetic field of the electromagnetic wave, we can neglect the latter. This means that the equation of motion for an electron in the plasma is

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v} \times \vec{B}. \tag{5.24}$$

Supposing that the wave propagates along the direction of the local magnetic field:

$$\vec{B} = B_0 \hat{\epsilon}_3, \tag{5.25}$$

with $\hat{\epsilon}_1 \times \hat{\epsilon}_2$.

Substitute this and the Equation (5.23) into the electron's equation of motion and solve for the velocity. To make the problem tractable, we look for a "steady state" solution, i.e. one which is oscillating rather than exponentially decaying. To achieve take the ansatz

$$\vec{v} = (\alpha + i\beta)\, e^{-i\omega' t} \hat{\epsilon}_1 + (\gamma + i\delta)\, e^{-i\omega' t} \hat{\epsilon}_2. \tag{5.26}$$

The strategy is to compute $\frac{d\vec{v}}{dt}$, then substitute this and $\vec{v}$ into the equation of motion.

The first observation, coming from equating the phases on both sides of the equation, is that the phases need to agree, $\omega = \omega'$. This leaves us with four equations, coming from the real and imaginary parts of the coefficients of $\hat{\epsilon}_1, \hat{\epsilon}_2$, in four unknowns $\alpha, \beta, \gamma, \delta$. Solving these equations leads to

$$\vec{v}(t) = -i\frac{e}{m_e\left(\omega \pm \omega_B\right)}\vec{E}(t), \tag{5.27}$$

where $\omega_B = \frac{eB_0}{m_e}$ is the gyration frequency of the electrons.

Different directions of polarisation for the electromagnetic wave thus result in different velocities for the electrons in the plasma. This leads to a different conductivity depending on the direction of polarisation:

$$\vec{j} = \sigma\vec{E} = -n_e e\vec{v} \tag{5.28}$$

$$= i\frac{n_e e^2}{m\left(\omega \pm \omega_B\right)}\vec{E}, \tag{5.29}$$

$$\Rightarrow \sigma = i\frac{n_e e^2}{m\left(\omega \pm \omega_B\right)} = i\frac{\varepsilon_0 \omega_{pe}^2}{\omega \pm \omega_B}. \tag{5.30}$$

The different conductivities in turn lead to different group velocities for the propagation of the wave through the plasma. To see this relate the conductivity to the permittivity[17],

$$\varepsilon_{R,L} = 1 - \frac{\sigma}{i\varepsilon_0\omega} = 1 - \frac{\omega_{pe}^2}{\omega\left(\omega \pm \omega_B\right)}. \tag{5.31}$$

[16] Checking that both expressions agree is a matter of expanding the complex exponential and looking at its real part. This was demonstrated in the lectures.

[17] The previous result for the permittivity is recovered for $B0 = 0$.

The phase vector that $\vec{E}$ rotates through for a given component travelling a distance ds through the plasma is kds. The total phase change after travelling a distance d through the plasma is

$$\phi_{R,L} = \int_0^d k_{R,L} ds. \tag{5.32}$$

The phase difference between the two components is thus

$$\Delta\theta = \phi_R - \phi_L = \int_0^d (k_R - k_L)\, ds. \tag{5.33}$$

To evaluate it further we make use if the phase velocity, $v_{\text{ph}} = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon}}$, which enables us to write

$$k_{R,L} = \frac{\omega}{c}\sqrt{\varepsilon_{R,L}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega\left(\omega \pm \omega_B\right)}}. \tag{5.34}$$

This can be simplified if we assume that the frequency of the electromagnetic wave is much greater than both the electron plasma frequency, and the electron gyration/cyclotron frequency. These approximations allow us to make two Taylor expansions, and write

$$k_{R,L} = \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega\left(\omega \pm \omega_B\right)}\right) \tag{5.35}$$

$$= \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\left(1 \pm \frac{\omega_B}{\omega}\right)^{-1}\right) \tag{5.36}$$

$$\simeq \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\left(1 \mp \frac{\omega_B}{\omega}\right)\right]. \tag{5.37}$$

The difference between the wave vectors is

$$k_R - k_L = \frac{\omega}{c}\frac{\omega_{pe}^2}{\omega^2}\frac{\omega_B}{\omega} \tag{5.38}$$

$$= \frac{\omega_B}{c}\frac{\omega_{pe}^2}{\omega^2} \tag{5.39}$$

$$= \frac{1}{\omega^2 c}\frac{n_e e^2}{m_e \varepsilon_0}\frac{eB_0}{m_e}, \tag{5.40}$$

which gives the phase difference as

$$\Delta\theta = \frac{e^3}{m^2 \varepsilon_0 c \omega^2}\int_0^d n_e B_0 ds. \tag{5.41}$$

The electron concentration and the magnetic field stay within the integral as they can vary along the path that the electromagnetic wave travels. The result can be generalised to account for the wave not travelling parallel to the direction of the magnetic field. If this is done $B_0 ds$ is replaced by $\vec{B} \cdot d\vec{s}$.

Faraday rotation is the effect that this phase difference between the LCP and RCP leads to a rotation in the direction of linear polarisation. To understand this we need to know the relationship between $\Delta\theta$ and $\Delta\chi$.

Add a figure

The relationship is that

$$\Delta\chi = \frac{1}{2}\Delta\theta. \tag{5.42}$$

Conventionally the change in the direction of linear polarisation is expressed in terms of the wavelength, $\lambda^2 = \left(\frac{2\pi c}{\omega}\right)^2$, rather than the frequency:

$$\Delta\chi = \frac{1}{2}\frac{e^3}{m^2\varepsilon_0 c}\left(\frac{\lambda}{2\pi c}\right)^2 \int_0^d n_e\vec{B}\cdot d\vec{s} = \frac{e^3}{8\pi^2 m_e^2\varepsilon_0 c^3}\left(\int_0^d n_e\vec{B}\cdot d\vec{s}\right)\lambda^2. \tag{5.43}$$

The coefficient of λ^2 is known as the rotation measure,

$$\mathrm{RM} = \frac{e^3}{8\pi^2 m_e^2\varepsilon_0 c^3}\left(\int_0^d n_e\vec{B}\cdot d\vec{s}\right). \tag{5.44}$$

The rotation measure is found by measuring the direction of linear polarisation at several wavelengths. While we can infer the value of $\int_0^d n_e\vec{B}\cdot d\vec{s}$, we cannot in general disentangle $n_e(s), \vec{B}(s)$ and the path taken.

Similar to the dispersion measure we can only infer typical or average values in most cases. Not, the how the values are distributed along the line of sight. If the sign of the rotation measure changes, this indicates that the magnetic field changes direction along the line of sight. The rotation measure gives a lower limit on the integral, since if the magnetic field reverses direction it reduces the size of the rotation measure.

If the wave is linearly polarised, for example if it is due to synchrotron radiation, the observed rotation measure can be used to give a $3D$ view of the magnetic field structure. This is because, synchrotron radiation is associated with the component of the magnetic field in the plane of the sky. While, the Farady rotation is associated with the component of the magnetic field along the line of sight.

5.4 Razin Effect

We now want to give an example of how passing through a plasma changes the radiation[18]. The angular range within which radiation is emitted by a relativistic charge is concentrated around the forward direction of motion:

$$\theta_b \sim \frac{1}{\gamma} = \sqrt{1-\beta^2}. \tag{5.45}$$

This is true for a source moving in a vacuum. In a medium the speed of light is nolonger c but $\frac{c}{n_r}$. Thus the angular range is

$$\theta_b \sim \sqrt{1-n_r^2\beta^2}, \tag{5.46}$$

with $n_r^2 = 1 - \frac{\omega_{pe}^2}{\omega^2}$ for a plasma.

The are two cases to consider:

[18]Another such effect, relevant when $n_r > 1$ is Cherenkov radiation. This is when the velocity of the charges exceeds the phase velocity of an electromagnetic wave in the medium. In a future version of these notes I may add a discussion of Cherenkov radiation.

- If the refractive index is of order 1, $n_r \sim 1$, this reduces to the vacuum case.
- For an ultra relativistic charge, $\beta \sim 1$, if the refractive index is very different from 1 the angular range is

$$\theta_b \sim \sqrt{1-n_r^2} \sim \sqrt{1-\varepsilon} \sim \frac{\omega_{pe}}{\omega}. \tag{5.47}$$

  Here the index of refraction dominates at low frequencies. As the frequency increases, θ_b decreases until it becomes $\sim \frac{1}{\gamma}$. e.g the medium is important when $\frac{\omega_{pe}}{\omega} \gg \frac{1}{\gamma}$.

The *Raizin effect* is that synchrotron radiation travelling through a medium gets cut off at $\omega \ll \gamma\omega_{pe}$. i.e the beaming effect is suppressed[19].

6 Absorption and Emission of radiation

We have spent most of the course discussing electromagnetic radiation. Now, we turn to the process of emission and absorption of this radiation.

The equations of radiative transfer, and the various quantities with in it, deal with macroscopic conditions and properties that have their origin in microscopic properties. To understand this better consider a system of 2 discrete energy levels, see Figure 10

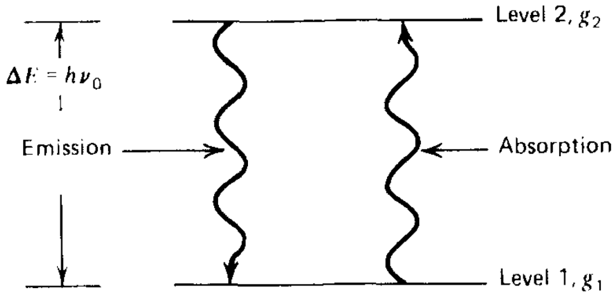

Figure 10: Taken from [1], a two level system with separation energy $\Delta E = h\nu_0$. The labels g_1 and g_2 are the statistical weights.

Einstein identified three processes connecting the two states.

1: Spontaneous Emission. This occurs when the system drops from energy level two to energy level one. It can occur in the absence of external radiation fields, and is described by the Einstein A coefficient:

$$A_{21} = \frac{\text{Transition probability for spontaneous emission}}{\text{unit time}}. \tag{6.1}$$

2: Absorption. The system absorbs a photon with energy $h\nu_0$ and transitions from level one to level two. This process requires an external radiation field. As photons do not interact the

[19]Note that the conditions $\omega \gg \omega_{pe}$ and $\omega \ll \gamma\omega_{pe}$ may look incompatible. However, for ultra relativistic particles it is possible for both to be satisfied.

transition probability is proportional to the mean intensity (photon density) at the frequency ν_0. The process is described by one of the Einstein B coefficients:

$$B_{12}\bar{J} = \frac{\text{Transition probability for absorption}}{\text{unit time}}, \tag{6.2}$$

with $\bar{J}$ the mean intensity at ν_0.

3: Stimulated Emission. Here the system transitions from level two to level one in the presence of a photon of energy $h\nu_0$. This requires an external radiation field, and consistency with the Planck black body law implies that the transition probability is proportional to the mean intensity. It is described by another Einstein B coefficient:

$$B_{21}\bar{J} = \frac{\text{Transition probability for stimulated emission}}{\text{unit time}}. \tag{6.3}$$

If the difference between the energy levels gives rise to a line in the spectrum (it cannot be infinitely sharp) this is described by the line profile function $\phi(\nu)$. The line profile function is peaked around $\nu = \nu_0$, and becomes a delta function in the infinitely sharp limit. It satisfies,

$$\int_0^\infty \phi(\nu)d\nu = 1, \tag{6.4}$$

and the mean intensity at ν_0 is computed from

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu)d\nu, \tag{6.5}$$

with J_ν the mean intensity.

There are some nice relationships between the Einstein coefficients. These are found by considering a system in thermodynamic equilibrium, where:

$$\frac{\text{Transitions out of level 1}}{\text{time Volume}} = \frac{\text{Transitions into level 1}}{\text{time Volume}}, \tag{6.6}$$

or

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \tag{6.7}$$

Here n_1, n_2 are respectively the number density (concentration) of atoms in level one and level two.

Some algebra enables us to solve this for $\bar{J}$,

$$\bar{J} = \frac{\frac{A_{21}}{B_{21}}}{\frac{n_1 B_{12}}{n_2 B_{21}} - 1}. \tag{6.8}$$

In thermodynamic equilibrium we know that

$$\frac{n_1}{n_2} = \frac{g_1 e^{-\frac{E_1}{k_B T}}}{g_2 e^{-\frac{E_2}{k_B T}}} = \frac{g_1}{g_2} e^{\frac{h\nu_0}{k_B T}}, \tag{6.9}$$

where E_1, E_2 are the energies of the two levels.

Thus the mean intensity is

$$\bar{J} = \frac{\frac{A_{21}}{B_{21}}}{\frac{g_1 B_{12}}{g_2 B_{21}} e^{\frac{h\nu_0}{k_B T}} - 1}. \tag{6.10}$$

However in thermal equilibrium this should be given by the black body distribution

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}. \tag{6.11}$$

Comparing Equations (6.10) and (6.11) we see that the Einstein coefficients are related through:

$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2} \qquad \Rightarrow A_{21} = \frac{2h\nu^3}{c^2} B_{21}, \tag{6.12}$$

$$\frac{g_1 B_{12}}{g_2 B_{21}} = 1, \qquad \Rightarrow g_1 B_{12} = g_2 B_{21}. \tag{6.13}$$

These relations do not depend on temperature, so they hold outside of equilibrium. Thus if we can determine one Einstein coefficient, we can determine the other two. Including stimulated emission is essential to get $\bar{J}$ to correspond to Planck's law. It turns out that the photon emitted during stimulated emission has the same direction, phase, and frequency as the photon that stimulated its emission.

Absorption and Emission coefficients. The absorption and emission coefficients can be expressed in terms of the Einstein coefficients. The emission coefficient is

$$j_\nu = \frac{\text{Energy emitted}}{\text{time solid angle}} = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu). \tag{6.14}$$

The first fraction is the energy per transition per solid angle, and the $n_2 A_{21}$ gives the number of transitions per time per volume.

We also know that

$$\alpha_\nu I_\nu = \frac{\text{Energy absorbed}}{\text{time solid angle volume}} = \frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu), \tag{6.15}$$

with $n_1 B_{12} I_\nu$ the number of transitions per volume per unit time.

Thus the absorption coefficient, uncorrected for the presence of stimulated emission, is

$$\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu). \tag{6.16}$$

To account for stimulated emission treat it as negative absorption. This implies that

$$\alpha_\nu = \frac{h\nu_0}{4\pi} \left(n_1 B_{12} - n_2 B_{21}\right) \phi(\nu). \tag{6.17}$$

Inverted Populations. We can use the relation $g_1 B_{12} = g_2 B_{21}$ in the absorption coefficient to get

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\left(n_1 B_{12} - n_2 \frac{g_1}{g_2} B_{12}\right)\phi(\nu) = \frac{h\nu_0}{4\pi} n_1 B_{12}\left[1 - \frac{g_1 n_2}{g_2 n_1}\right]\phi(\nu). \tag{6.18}$$

In thermal equilibrium the ratio of number densities is

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\frac{h\nu_0}{k_B T}}, \tag{6.19}$$

which implies that

$$\frac{g_1 n_2}{g_2 n_1} = e^{-\frac{h\nu_0}{k_B T}} < 1, \tag{6.20}$$

and the absorption coefficient is positive. This is refereed to as the system having a normal population.

If there is an inverted population, e.g. more atoms in the upper level than the lower level, then

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}. \tag{6.21}$$

This implies that

$$\frac{g_1 n_2}{g_2 n_1} > 1, \tag{6.22}$$

and the absorption coefficient is negative. We said above that negative absorption corresponds to stimulated emission. This creates a situation where the intensity increases along a ray, and leads to the concept of a *maser*[20] and a *laser*. The frequency range of the physical processes determines where the emitted photons are on the electromagnetic spectrum.

A Extra Examples

These are some extra examples, some of which I will present in the tutorial sessions. The solutions will start to appear after the tutorial sessions.

Example A.1. Problem 9.9 **from [2]**
Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is

(a) travelling in the negative x-direction and polarised in the z-direction,

(b) travelling in the direction from the origin to the point $(1, 1, 1)$, with polarisation parallel to the xz plane.

Solution A.1. *The solution to this is given in the slides from Tutorial 1.*

Example A.2. Problem 9.10 **from [2]**
The intensity of sunlight hitting the earth is about $1300 W m^{-2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

[20]Microwave amplification by stimulated emission of radiation.

Solution A.2. *The intensity of light hitting the earth is* $I = 1300Wm^{-2}$.

- *The radiation pressure on a perfect absorber is related to the intensity through* $P = \frac{I}{c} = 4.3 \times 10^{-6}W$.
- *For a perfect reflector, we get double the radiation pressure as the reflected radiation exerts an equal pressure through Newton's second law. This means that* $P = 8.6 \times 10^{-6}W$.

Example A.3. Problem 9.33 **from [2]**
Suppose

$$\vec{E} = A\frac{\sin\theta}{r}\left[\cos\left(kr - \omega t\right) - \frac{1}{kr}\sin\left(kr - \omega t\right)\right]\hat{\varphi}, \quad \text{with } \frac{\omega}{k} = c. \tag{A.1}$$

This is the simplest possible *spherical wave.*

(a) Show that $\vec{E}$ obeys Maxwell's equations in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. Does $\vec{I}$ point in the expected direction? Does it fall of like $\frac{1}{r^2}$, as it should?

(c) Integrated $\vec{I} \cdot d\vec{a}$ over a spherical surface to determine the total power radiated.

B Problems

I will include exercises here and use them for the problem sheets. Both [1] and [2] include lots of problems so I am likely to use some of those. This will be split up into subsections corresponding to the topics.

Problem B.1. R&L Problem 2.1 *For two oscillating quantities* $A(t)$ *and* $B(t)$*, the real parts of* $\mathcal{A}e^{-i\omega t}$ *and* $\mathcal{B}e^{-i\omega t}$ *with* $\mathcal{A}$ *and* $\mathcal{B}$ *complex, show that the time average is given by*

$$\langle AB \rangle = \frac{1}{2}Re\left(\mathcal{A}\mathcal{B}^*\right) \tag{B.1}$$

Hint: The time average is give by $\langle A \rangle = \frac{1}{T}\int_0^T A(t)dt$ *where* T *is the period of* A.

Problem B.2. Telegraph equation

(a) Show that the solutions to the Telegrapher's equation,

$$-\frac{\partial^2}{\partial x^2}\vec{E} + \mu\sigma\frac{\partial}{\partial t}\vec{E} + \mu\varepsilon\frac{\partial^2}{\partial t^2}\vec{E} = 0, \tag{B.2}$$

are damped waves.

(b) Calculate the complex index of refraction defined through

$$k^2 = \frac{\omega^2 n^2}{c^2}, \qquad n = n' + in'', \quad n', n'' \in \mathbb{R}. \tag{B.3}$$

The vacuum speed of light is $c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \simeq 3 \times 10^8$.

Problem B.3. Fourier Transform
Calculate the Fourier transform of a Gaussian pulse with width (standard deviation) σ*,*

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}. \tag{B.4}$$

Show that the Fourier transform is again a Gaussian pulse with width $\tilde{\sigma}$*, and that* $\sigma\tilde{\sigma}$ *is a constant.*

Hint: Try expressing $t^2 + i\omega t$ *as* $\left(a + (t + ib)^2\right)$ *for some* a, b*. Also recall the Gaussian integral* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$*.*

Problem B.4. Parseval's Theorem *Show Parseval's Theorem:*

$$\int_{-\infty}^{\infty} |E(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega. \tag{B.5}$$

This equality means that the total energy of a signal can be determined either by integrating the power over time, or by integrating its spectrum over all frequencies.

Problem B.5. Radiation fields
Given the Liénard–Wiechert expression for the vector potential in Eq. (3.14) *that*

$$\vec{A}(\vec{r}, t) = \frac{q\mu_0}{4\pi} \frac{\vec{u}(t_r)}{R(t_r)\kappa(t_R)} = \frac{\vec{u}}{c^2}\phi(\vec{r}, t), \tag{B.6}$$

show that the magnetic field is given by

$$\vec{B} = \frac{\hat{n}}{c} \times \vec{E}. \tag{B.7}$$

Then identify the radiation field component of $\vec{B}$*. Note that we are using the same notation as in Section. 3,* t_r *is the retarded time,* $\vec{R} = \vec{r} - \vec{r}_0(t_r)$*,* $\hat{n} = \frac{\vec{R}}{R}$*,* $u(t_r) = \dot{\vec{r}}_0(t_r)$*, and* $\kappa = 1 - \hat{n} \cdot \frac{\vec{u}}{c}$*.*

Problem B.6. Angle of maximum emission
Consider an accelerating charged particle with $\vec{a}$ *parallel to the velocity. Given the angular dependence of the emitted power from Equation* (3.47)*, find the angle at which the emitted power is maximal.*

Problem B.7. Classical Bohr model, Griffiths Problem 11.14
In the Bohr model of the hydrogen atom the electron in its ground state is supposed to travel in a circle of radius $r_e = 5 \times 10^{-11} m$*, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron radiates, and hence will spiral in to the nucleus.*

(a) Show that $v_e \ll c$ *for most of the trip.*

(b) Use the Larmor formula, Equation (3.35) *to calculate the lifespan of the atom. Hint:Assume that the electron has a circular orbit.*

Problem B.8. Acceleration in different frames

(a) *Show that the transformation of acceleration between frames is given by*

$$a_x = \frac{a'_x}{\gamma^3\left(1+\frac{vu'_x}{c^2}\right)^3},$$
$$a_y = \frac{a'_y}{\gamma^2\left(1+\frac{vu'_x}{c^2}\right)^2} - \frac{u'_y v}{c^2}\frac{a'_x}{\gamma^2\left(1+\frac{vu'_x}{c^2}\right)^3},$$
$$a_z = \frac{a'_z}{\gamma^2\left(1+\frac{vu'_x}{c^2}\right)^2} - \frac{u'_z v}{c^2}\frac{a'_x}{\gamma^2\left(1+\frac{vu'_x}{c^2}\right)^3}.$$

Here v the relative velocity between the frames is in the x-direction, (u'_x, u'_y, u'_z) are the velocity components of an object in the primed frame, (a'_x, a'_y, a'_z) are the components of its acceleration, and (a_x, a_y, a_z) are the components of the acceleration in the unprimed frame. Note: you have solved a similar problem in PY2106 when you considered the velocity transformation equations.

(b) *Now suppose that the primed frame is the instantaneous rest frame of a particle. Show that in this case*

$$a'_{\parallel} = \gamma^3 a_{\parallel}$$
$$a'_{\perp} = \gamma^2 a_{\perp}$$

where $\parallel$ and $\perp$ refer to the components parallel to and perpendicular to the direction of v, respectively.

Problem B.9. *Ultra relativistic Electrons*
Consider an ultra relativistic electron emitting synchrotron radiation.

(a) *Show that the energy of the charge decreases with time according to*

$$\gamma = \frac{\gamma_0}{1 + A\gamma_0 t}, \tag{B.8}$$

with γ_0 the initial value of γ, and find the constant A.

(b) *Show that the time for the electron to lose half its energy is*

$$t_{\frac{1}{2}} = \frac{1}{A\gamma_0} \tag{B.9}$$

(c) *How is the decrease in γ implied here reconciled with the fact that the equation of motio implied that γ is constant?*

C Vector Calculus Review

There is a nice vector calculus review in [2]. My review is taken from the PY2101 Classical mechanics notes.

C.1 Cartesian coordinates

The most familiar coordinate system to everyone is the old familiar Cartesian coordinates with unit vectors $\hat{x}, \hat{y}, \hat{z}$ that do not depend on position. In Cartesian coordinates the position vector is shown in Fig. 11 and is given by

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{C.1}$$

The derivative of the position vector is

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \tag{C.2}$$

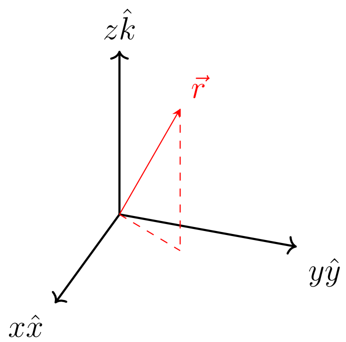

Figure 11: Cartesian Coordinate system

The key object in vector clculus is the gradient "vector" ∇ given by

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}. \tag{C.3}$$

This differential operator is called Grad or nabla and can act on scalars and vectors. The action on scalars gives the gradient of the scalar function, a measure of how the function changes in each direction.

$$\nabla f(x, y, z) = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \tag{C.4}$$

This is related to the usual expression for the gradient of a function in the following way

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f \cdot d\vec{r}. \tag{C.5}$$

As ∇ is treated like a vector it can act on vectors through both the scalar and vector. These are:

1. The Divergence

$$\nabla \cdot \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{k}\right),$$
$$= \hat{x} \cdot \frac{\partial \vec{A}}{\partial x} + \hat{y} \cdot \frac{\partial \vec{A}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{A}}{\partial z},$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

   Where we have used that the unit vectors do not depend on the coordinates.

2. The curl, which is easiest to express as a the determinant of a matrix

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{C.6}$$

There are many useful identities about the interaction of ∇, $\nabla\cdot$, $\nabla\times$ and their action on products of vectors, these include:

$$\nabla \cdot (\nabla f) = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{C.7}$$
$$\nabla \times (\nabla f) = 0, \tag{C.8}$$
$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0, \tag{C.9}$$
$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}, \tag{C.10}$$
$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B), \tag{C.11}$$
$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla) A - (A \cdot \nabla) B, \tag{C.12}$$

In the above $\nabla^2\vec{A}$ is the vector Laplacian, in general it is defined through this relation but in Cartesian coordinates it is given by $\nabla^2\vec{A} = \hat{x}\Delta A_x + \hat{y}\Delta A_y + \hat{k}\Delta A_z$.

These operations all become more complicated in other coordinate systems. The two that are often useful to consider are cylindrical (ρ, φ, z) and spherical (r, θ, ϕ)

C.2 Cylindrical coordinates

The coordinates directions are now (ρ, φ, z) with unit vectors $\hat{z}, \hat{\rho}, \hat{\varphi}$ that are no longer all position independent. The coordinate system and the position vector are shown in Fig. 12. The derivative of the position vector is now

$$d\vec{r} = dr_\rho\, \hat{\rho} + dr_\varphi\, \hat{\varphi} + dr_z\, \hat{z} \tag{C.13}$$

Thinking about infinitesimal changes in the position vector we have that $dr_\rho = d\rho$ and $dr_z = dz$. However, $dr_\varphi = \rho d\varphi$. The volume element is $dV = \rho d\rho d\varphi dz$. To see this draw the arc of a circle with radius ρ and angle $d\varphi$ the length of the arc is then $rd\varphi$ using the definition of an angle in radians.

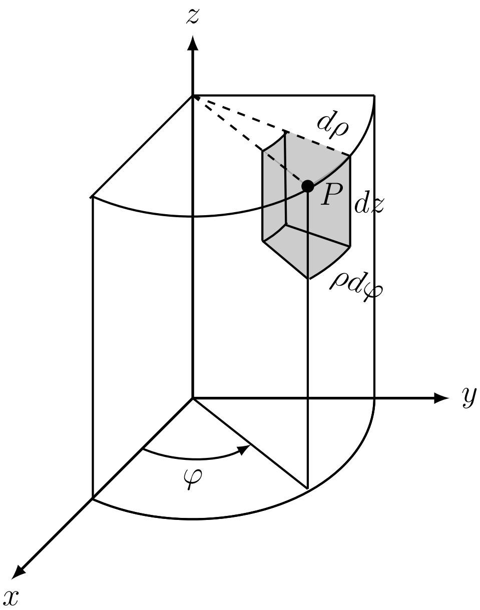

Figure 12: Cylindrical coordinate system

To compute Grad in cylindrical coordinates go back to $d\vec{f} = \nabla f \cdot d\vec{r}$, expanding both sides leads to

$$df = (\nabla f)_\rho \, d\rho + (\nabla f)_\varphi \, \rho d\varphi + (\nabla f)_z \, dz, \tag{C.14}$$

$$df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \varphi} d\varphi + \frac{\partial f}{\partial z} dz. \tag{C.15}$$

Comparing these two expressions gives that

$$\nabla = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}. \tag{C.16}$$

As $\hat{\rho}$ and $\hat{\varphi}$ both depend on φ care is needed when acting on vectors. To evaluate $\nabla\cdot$ and $\nabla\times$ them the relation between cylindrical unit vectors and Cartesian unit vectors is needed:

$$\hat{\rho} = \hat{x}\cos\varphi + \hat{y}\sin\varphi, \tag{C.17}$$

$$\hat{z} = z, \tag{C.18}$$

$$\hat{\varphi} = \hat{z} \times \hat{\rho} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi. \tag{C.19}$$

Thus $\frac{\partial \hat{\rho}}{\partial \varphi}, \frac{\partial \hat{\varphi}}{\partial \varphi} \neq 0$. Taking account of this leads to

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}. \tag{C.20}$$

Similarly evaluating the curl of a vector in cylindrical coordinates is harder. Going through the work leads to

$$\begin{aligned} \nabla \times \vec{A} =& \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\varphi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) \\ &+ \hat{z}\frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right). \end{aligned} \tag{C.21}$$

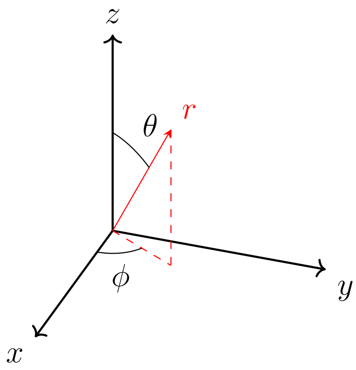

Figure 13: Spherical coordinate system

C.3 Spherical coordinates

Spherical coordinates are (r, θ, ϕ), with unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ which again are position dependent. The position vector shown in Fig. 13 is $\vec{r} = r\hat{r}$ and its derivative is

$$d\vec{r} = dr_r\,\hat{r} + dr_\theta\,\hat{\theta} + dr_\phi\,\hat{\phi} \tag{C.22}$$

Again thinking about infinitesimals gives $dr_r = dr$ and $dr_\theta = rd\theta$, and $dr_\phi = r\sin\theta d\phi$. The volume element is $dV = r^2 \sin\theta dr d\theta d\phi$.

The gradient vector is

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{C.23}$$

The unit vectors are related through

$$\hat{r} = \hat{\imath}\cos\theta\sin\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\phi, \tag{C.24}$$

$$\hat{\theta} = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\sin\theta\cos\phi - \hat{k}\sin\phi, \tag{C.25}$$

$$\hat{\phi} = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi \tag{C.26}$$

This leads to a θ and ϕ dependence. The expressions for $\nabla\cdot$ and $\nabla\times$ are much more complicated:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \tag{C.27}$$

$$\nabla\times\vec{A} = \left(\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right), \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}, \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\right) \tag{C.28}$$

If any other expressions are needed then this appendix will be extended.

C.4 Integral Theorems

Armed with an understanding of Gradient, Divergence, and Curl we can now state the two most important integral theorems that we will need from time to time.

1. Gauss's Law/ the Divergence Theorem

$$\oint_{\partial V} \vec{A} \cdot d\vec{a} = \int_V \left(\nabla \cdot \vec{A}\right) dV. \tag{C.29}$$

   Here V is a volume with boundary, ∂V, and $d\vec{a}$ is the normal area element.

2. Stoke's Law

$$\oint_{\partial S} \vec{A} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{A}\right) \cdot d\vec{a}. \tag{C.30}$$

   Here S is a surface with normal area element $d\vec{a}$ and boundary ∂S.

D A primer on the Fourier transform

The Fourier transform is a ubiquitous tool in physics and mathematics, it give a decomposition of a function into eigenfunctions of the Laplacian. The most familiar example is probably Fourier series in $1D$ where periodic functions are expressed as the sum of exponential functions at different frequencies, e.g. for $f(t)$ a function with periodicity T the Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\Omega t}, \quad \Omega = \frac{2\pi}{T} \text{ the angular frequency.} \tag{D.1}$$

The coefficients are given by an integral over $f(t)$ as is found by considering

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{ik\Omega t} dt = \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(k-n)\Omega t} dt = \sum_{n=-\infty}^{\infty} c_n T \delta_{k,n} = T c_k. \tag{D.2}$$

For non-periodic functions this needs to be generalised to the case of continuous frequencies. This leads to the Fourier transform and it's inverse:

$$F(\omega) = \int_{-\infty}^{\infty} f(t') e^{i\omega t'} dt', \tag{D.3}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega. \tag{D.4}$$

The continuous frequency variable is ω and the conventions for where to put the factor of 2π differ in different places. Often there will be a $\frac{1}{\sqrt{2\pi}}$ in both terms. However, it can be more convenient to put the factor entirely in the inverse Fourier transform. Note that for our purposes the Fourier transform only exists for square integrable functions, $f(t)$ such that $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$.

Example D.1. Single square pulse with width τ and height $\frac{1}{\tau}$.
Find the Fourier transform of the signal in Figure 14

The function is defined piecewise as

$$f(t) = \begin{cases} \frac{1}{\tau}, & t \in (-\frac{\tau}{2}, \frac{\tau}{2}) \\ 0, & \text{else} \end{cases} \tag{D.5}$$

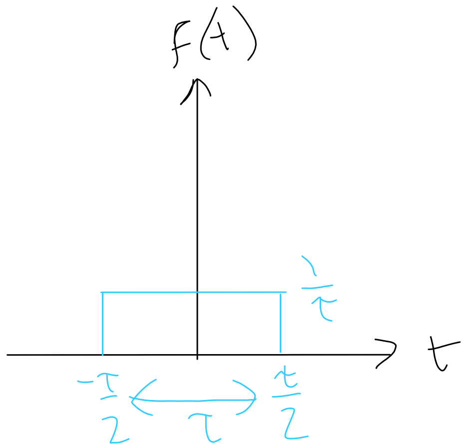

Figure 14: Square wave signal

so the Fourier transform is

$$\begin{aligned} F(\omega) &= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{i\omega t} dt \\ &= \frac{1}{\tau}\frac{1}{i\omega}\left(e^{i\omega t}\right)_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\ &= \frac{1}{i\omega\tau}\left(e^{i\frac{\omega\tau}{2}} - e^{-i\frac{\omega\tau}{2}}\right) \\ &= \frac{2}{\omega\tau}\sin\left(\frac{\omega\tau}{2}\right). \end{aligned}$$

The function $\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$ is known as the Sinc function and has a peak at $\omega = 0$. Looking at where $F(\omega) = 0$ we find that the width of $F(\omega)$ is given by $\frac{4\pi}{\tau} = \frac{4\pi}{\Delta t}$. This leads to the time-frequency uncertainty principle

$$\Delta\omega\Delta t = 4\pi. \tag{D.6}$$

Contrast this with the uncertainty principle from Quantum mechanics, $\Delta x \Delta p \geq \frac{\hbar}{2}$.

There is a spatial version of the Fourier transform found by replacing t with the spatial coordinates and ω with the wave vector $\vec{k}$. This leads to the d-dimensional Fourier transform:

$$f(t) = \frac{1}{(2\pi)^d}\int d^d\vec{k} F(\vec{k}) e^{i\vec{k}\cdot\vec{r}}, \quad F(\vec{k}) = \int d^d r f(\vec{r}) e^{-i\vec{k}\cdot\vec{r}}, \tag{D.7}$$

Note that it is conventional to have the opposite signs in the exponential factors relative to the time Fourier transform.

References

[1] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. 1986.

[2] David J Griffiths. *Introduction to electrodynamics; 4th ed.* Pearson, Boston, MA, 2013. Re-published by Cambridge University Press in 2017.

[3] M.P. Hobson, E.G. P, G.P. Efstathiou, and A.N. Lasenby. *General Relativity: An Introduction for Physicists*. General Relativity: An Introduction for Physicists. Cambridge University Press, 2006.

[4] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Butterworth-Heinemann, 4 edition, January 1980.

[5] John David Jackson. *Classical electrodynamics; 2nd ed.* Wiley, New York, NY, 1975.