YM on RS week 5

Equivariant morse theory recap:

$f : M \longrightarrow \mathbb{R}$

$G \circlearrowright M$

if action free $f_G : M/G \to \mathbb{R}$

f G-invariant

- critical pts of f $\rightsquigarrow$ critical submflds of f_G $N = G/H$ contribute $t^{\lambda(N)} P_t(BH)$

morse discussion lifts to

$Ex_G M$ universal Bundle $= M_G$

$f_G : M_G \to \mathbb{R}$ $G = G/G_0$

$M_G \simeq M_{\mathcal{G}}$

YM "$M = \mathcal{A}$"

$\mathcal{G}$ = $\mathfrak{g}$ valued fns $C^\infty(M, \mathfrak{g})$ $\rightsquigarrow$ $\mathcal{G}$ gauge group non free action

- critical submfld of f N $\rightsquigarrow$ contribute $t^{\lambda(N)} \underbrace{P_t(N_G)}$

refined as $\sum_i t^i \dim H^i(N_G)$ $= \sum_i t^i \dim H^i_G(N) = GP_t(N)$

$\nu^-(N)$ not orientable

$\rightsquigarrow$ work locally

$\mathcal{M}_t(M_G) = P_t(M_G)$ f Equivariantly perfect if

$\forall$ critical sub mflds, N, $e(\underbrace{\mathcal{N}(N)}_{\text{normal bundle to } N})$ (which is equivariant)

Euler class

if $\mathcal{N}(N)$ non orientable this is not a global section

is not a zero divisor in $H^*_G(N, K)$ $\forall K$

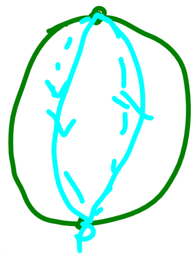
Geometric aspects of morse theory

- If Riemannian metric on M then define ∇f, dual to df,
- "gradient flow" of f is then given by the path of steepest descent i.e trajectories of $-\text{grad} f$ "integral curves"

- If f has only non-degenerate critical pts then every trajectory converges to some p ← critical point

• The set of all trajectories converging to p is form a cell $M^+(p)$

max



min

$M^+(p)$ 2-cell

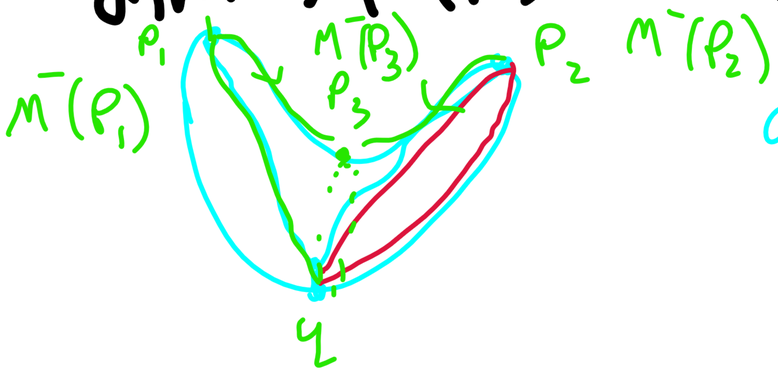
paths flowing down to p.

stable mfld of p since p absolute minima of $f|_{M^+(p)}$

• $f \longrightarrow -f$ $\quad M^+(p) \longrightarrow M^-(p)$

unstable mfld of p

$$\dim M^-(p) = \operatorname{codim}(M^+(p)) = \lambda_p$$



$q \in M^+(p), M^-(p)$

$T_q M^+(p)$ & $T_q M^-(p)$ span $T_q M$

$u(t)$ $\quad u(0) = r$

$\lim_{t\to-\infty} u(t) = p_1$

$\lim_{t\to\infty} u(t) = q$

$$\frac{du}{dt} = -\nabla f(u(t))$$

in QCD p, q are vacuna (saddle points)

← Energy $\int |\nabla f|^2 d^4x$

• $M = \bigcup_p M^+(p)$ Cell decomposition

morse inequalities follow from cellular homology of these cells!

There is usually a transversality condition

s.t. for $q \in M^-(p), M^+(p)$

$$T_q M^-(p) \ \& \ T_q M^+(p) \text{ span } T_q M$$

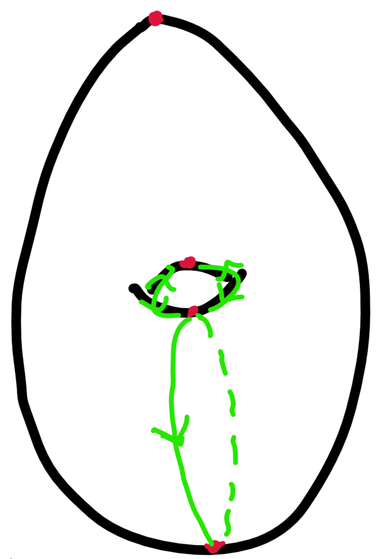
Not true in our favourite example:

max — p

saddle — s_1

saddle — s_2

min — q



$M^-(p)$ 2-cell

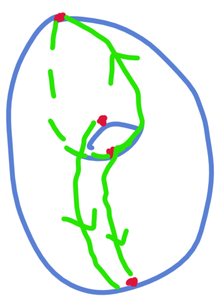
$M^-(s_1)$ 1-cell

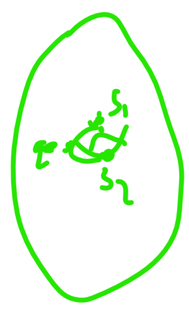
$M^-(s_2)$ 1-cell

$M^+(s_2) = M^-(s_1)$ so violates transversality

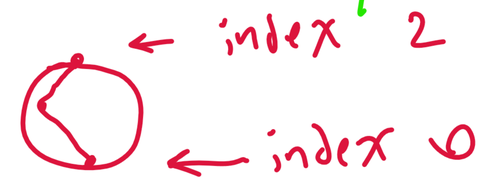
$M^-(q)$ 0-cell

perturb $-\nabla f$ to fix this





$$T_q M \neq T_q M^-(s_1) \oplus T_q M^+(s_2)$$



Non-degenerate critical sub mflds $\rightsquigarrow$ stable mflds

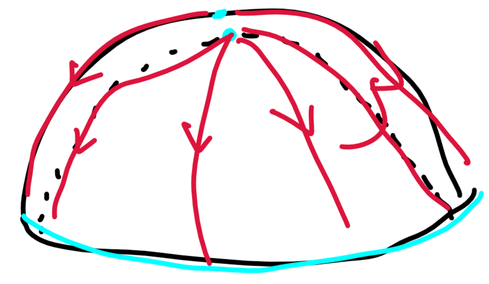
N

$M^+(N)$

cell-bundles over N

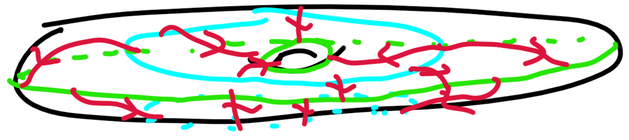
whatever this means?

"$\mathbb{RP}^2$"



e.g.

torus on its side



- max
- saddle
- min

$\rightsquigarrow$ morse stratification $\quad M = \bigcup_N M^+(N)$

<u>prop 1.12</u>: Let N_0 be minimum mfld of f and assume morse index of other critical sub mflds ≥ 3. Then if M is connected $\Rightarrow N_0$ is connected & $\pi_1(N_0) \cong \pi_1(M)$.

Equivariant case: f G-invariant,

$\rightsquigarrow$ G invariant metric on M_G ↑ cpct Lie group

- gradient flow is G-invariant so stratification is G-invariant e.g G action respects strata!

prop 1.12 has an equivalent equavariant version

Natural partial ordering of critical sub mflds $\prec$

$$N_1 \prec N_2 \iff \partial M^+(N_1) \cap M^+(N_2) \neq \emptyset$$

e.g $N_1 \prec N_2 \Rightarrow$ trajectory of $-\nabla f$ goes from N_1 to within ε of N_2 $(\varepsilon > 0)$

if $\varepsilon < f(N_2) - f(N_1)$

then $N_1 \prec N_2 \Rightarrow f(N_1) < f(N_2)$

↑ transitive partial ordering

gives $\overline{M^+(N)} = \bigcup_{\substack{N' \\ \text{st } f(N') \geq f(N)}} M^+(N')$

Some times write $M = \bigcup_\lambda M_\lambda$

locally closed sub mfld

index set of morse indices λ

strict partial ordering

so $\overline{M}_\lambda = \bigcup_{\mu \geq \lambda} M_\mu$

$\rightsquigarrow$ use the morse stratification to get morse type info about the homology of M How?

$\rightsquigarrow$ Do this inductively

start with open strata minimal λ then add other strata

$\rightsquigarrow$ $I \subset$ index set open if $\lambda \in I$ & $\mu \leq \lambda \Rightarrow \mu \in I$

closed if $\lambda \in I$ & $\mu \geq \lambda \Rightarrow \mu \in I$

I closed iff complement is open.

$M_I = \bigcup_{\lambda \in I} M_\lambda$ open (closed) if I is open (closed)!

If I open & $\lambda \in I'$ minimal then $J = I \cup \lambda$ open

$\rightsquigarrow$ this is the inductive step

$$M_\lambda = M_J - M_I \underset{\text{closed}}{\subset} M_J$$

- pass to K-cohomology (if $N(M_\lambda)$ orientable) $k = k_\lambda = \operatorname{codim} M_\lambda$

$$\cdots \to H^{q-k}(M_\lambda) \to H^q(M_J) \to H^q(M_I) \to \cdots$$

it breaks into short exact sequence $\forall_{I,\lambda}$

$$\Rightarrow \quad P_t(M) = \sum t^{k_\lambda} P_t(M_\lambda)$$

If true $\forall K$ stratification is perfect!

Perfect morse fn defines a perfect stratification!