Introduction to equivariant cohomology

G Lie group

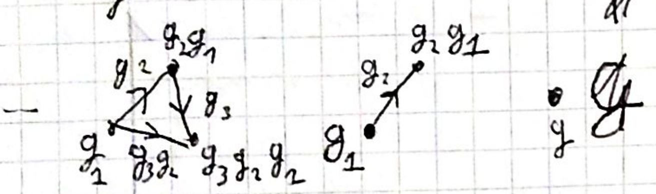
EG ~~free~~ contractible space with free G action

$BG := EG/G$

e.g. $EU(1) = S^\infty$ $\quad BU(1) = \mathbb{C}P^\infty$ $\quad \mathbb{Z}_2 \hookrightarrow U(1)$

$E\mathbb{Z}_2 = S^\infty$ $\quad B\mathbb{Z}_2 = \mathbb{R}P^\infty$

general construction



$G^3\times\Delta^2$ $\quad G^2\times[0,1]$ $\quad G$

$X \circlearrowleft G$ Why is "$H_G(X) := X/G$" not a good definition?

Consider a equivariant $f: X \to Y$ map which is a homotopy equivalence e.g.

$G \circlearrowleft EG \to * \circlearrowleft G$, but $H(BG) \neq H(*)$

For $X \circlearrowleft G$ we define $X_G = EG \times_G X$

$= \{(e,x) \in EG \times X\}/(e.g, x) \sim (e, g.x)$

$X \to X_G \to BG$

Define $H_G(X) = H(X_G)$

Assume action is free

$$EG \to EG \times_G X \xrightarrow{\pi} X/G$$

$\pi^{-1}([x]) = \{[(e,\tilde{x})] \mid \text{s.t. } \tilde{x} \in [x]\} = \{(e,x) \mid e \in EG\}$

$= EG$ $\Rightarrow$ π is a homotopy equivalence

$H_G(X) = H(X/G)$

$K \subset G$ closed normal subgroup $\quad X \circlearrowleft G$, s.t. K acts freely + $X \to X/K = Y$ is K-bundle

$S = G/K$

Prop $H_G(X) = H_S(Y)$

Proof $\quad E_G \times E_S$ has a G action via $G \to S$

$\Rightarrow X_G = X \times_G (E_G \times E_S) \xrightarrow{\simeq} X \times_G E_S \cong Y \times_S E_S = Y_S$ $\square$

G acts freely on $X \times E_S$

In application

$\mathcal{G}$ gauge group $\circlearrowright \mathcal{A}$ ← space of connections

~~$\mathcal{G}_0$ based~~ $\mathcal{G}_0 \subseteq \mathcal{G}$ gauge transformations which are ~~fixed at~~ trivial at a point

$\mathcal{G}/\mathcal{G}_0 = G$

$H_{\mathcal{G}}(\mathcal{A}) \cong H_G(\mathcal{A}/\mathcal{G}_0)$

Extension

$K \subset G$ $K \circlearrowright Y$ construct a G space by

$X = G \times_K Y$

$$X_G = EG \times_G X = EG \times_G G \times_K Y = \overset{EK}{EG} \times_K Y$$

$= Y_K$

$\Rightarrow H_G(X) = H_K(Y)$

Now we consider more specific case of the compact Lie groups with no torsion

e.g $G = U(n_1) \times \dots U(n_r)$

$H^*(U(n))$ is a polynomial in $c_1, \dots, c_n$ of degree $2i$.

Let $T \subset G$ be a maximal torus

$G/T \to BT \to BG$ behaves like a product in cohomology

$\Rightarrow$ The same for

$G/T \to X_T \to X_G$ with total grading

$H(X_T) = H(X_G) \otimes H(G/T)$

$\Rightarrow H(X_G)$ is a direct summand in $H(X_T)$

$\Leftrightarrow H_G(X, \mathbb{Z}_p) \to H_T(X, \mathbb{Z}_p)$ is injective for all primes p

Now $T = T_0 \times T_1$ with T_0 acting trivial

$\Rightarrow X_T = BT_0 \times X_{T_1}$

$\Rightarrow$ for $\mathbb{Z}_p$ coefficients $H_T(X) = H(BT_0) \otimes H_{T_1}(X)$

This a field — This ~~...~~ is polynomial

$H(BT_0)$ is a polynomial algebra

$\Rightarrow \alpha_0 \in H(BT_0)$ is no zero divisor

$\alpha_0 \neq 0$

More general

$\alpha \in H_T(X) \quad \alpha = \alpha_0 \otimes 1 + \underbrace{\ldots}_{\text{terms higher in } H(X_T)}$

has the same property

~~Proof~~: Consider the filtration

~~$H_0 = H(BT_0) \otimes 1 \subseteq H_1 = H(BT_0) \otimes H(X_T)^{\leq 1} \subseteq H_2 \subseteq \ldots$~~

$H_T(\cancel{BT_0}) = H(BT_0) \otimes H_{T_1}(X)^{\geq 0} \supseteq H(BT_0) \otimes H_{T_1}(X)^{\geq 1} \supseteq \ldots$

gr $H(BT) \otimes H_{T_1}(X)^i$ on which α acts by α_0 $\square$

α_0 will be the Chern class of a vector bundle $N_{T_0} \to BT_0$. For line bundles

~~BT~~ We get an iso $\mathrm{LBun}(BT) \longrightarrow H^2(BT_0, \mathbb{Z})$

$N \longmapsto c_1 N$

$BT \to BU(1)$ $\quad \cong \quad \hat{T}$

$\Leftrightarrow$ group homo $T \to U(1)$

$\mathrm{Hom}(T, U(1)) =$ character lattice

More general $H^*(BT_0, \mathbb{Z})$ is the symmetric algebra of the lattice $\hat{T}_0$

~~M~~ For an n-dimensional representation we can decompose

$$N = \sum_{j=1}^{n} L_j$$

$$c_n(N_{T_0}) = \prod_{j=1}^{n} c_j(L_j)$$

L_j is primitive if for all prim p

$\nexists$ $L_j/p \in \hat{T}$, s.t. $p\, L_j/p = L_j$

If all L_j are primitive $\Rightarrow$ the mod p-reductions of $c_n(N_{T_0})$ is non zero

---

This combines into

$X \circlearrowleft G \supseteq T_0$ acting trivial

N on X a G-vector (equivariant) bundle ~~(associated to a G-bun…)~~

T_0 acts primitive on the fibre $H(G)$ has no torsion $\Longrightarrow$ $\alpha = c_n(N_G)$ acts injectively on $H_G(X, \mathbb{Z}_p)$ for all p

Proof: Use 13.3 to restrict to tori and use the associated bundle construction