Twistor Theory Lectures: solutions to selected exercises

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I will try and provide solutions to some of the exercises that we were given in the lectures.

1 Conventions and useful facts

We will be interested in line bundles on $\mathbb{P}^1$ of degree m so it is useful to set out what we mean by the tautological and hyperplane bundle and their tensor powers.

Example 1.1. The tautological line bundle over $\mathbb{P}^1$ is denoted $\mathcal{O}(-1)$. In [1] the definition is

$$\mathcal{O}(-1) = \{(\lambda_{\dot{\alpha}}, \rho_{\dot{\alpha}}) \in \mathbb{P}^1 \times \mathbb{C}^2 | \rho_{\dot{\alpha}} = \mu\lambda_{\dot{\alpha}}, \mu \in \mathbb{C}\}, \tag{1.1}$$

In words we have that $\rho_{\dot{\alpha}}$ is an element of the complex line which passes through $\lambda_{\dot{\alpha}}$. Following [2] on $\mathbb{P}^1$ we use the two charts

$$U_+ = \{[\lambda_{\dot{1}} : \lambda_{\dot{2}}] | \lambda_{\dot{1}} \neq 0\}, \quad \phi_+ : [\lambda_{\dot{1}} : \lambda_{\dot{2}}] \mapsto \frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}} = \lambda_+,$$

$$U_- = \{[\lambda_{\dot{1}} : \lambda_{\dot{2}}] | \lambda_{\dot{2}} \neq 0\}, \quad \phi_- : [\lambda_{\dot{1}} : \lambda_{\dot{2}}] \mapsto \frac{\lambda_{\dot{1}}}{\lambda_{\dot{2}}} = \lambda_-.$$

We can immediately see that on the overlap $U_+ \cap U_-$ we have that $\lambda_+ = \lambda_-^{-1}$. We can immediately compute that the map $\psi_{+-} = \phi_+ \circ \phi_-^{-1} : \phi_-(U_+ \cap U_-) \to \phi_+(U_+ \cap U_-)$ satisfies

$$\psi_{+-}(\lambda_-) = \phi_+ \circ \phi_-^{-1} \circ \phi_-([\lambda_{\dot{1}} : \lambda_{\dot{2}}]) = \phi_0([\lambda_{\dot{1}} : \lambda_{\dot{2}}]) = \lambda_+. \tag{1.2}$$

To understand the bundle $\mathcal{O}(-1)$ we consider the two trivialisations

$$t_+ : \mathcal{O}(-1)|_{U_+} \to U_+ \times \mathbb{C},$$
$$t_- : \mathcal{O}(-1)|_{U_-} \to U_- \times \mathbb{C}.$$

More explicitly the maps act as follows

$$t_+ : ([1 : \lambda_+], \mu(\lambda_{\dot{1}}, \lambda_{\dot{2}})) \mapsto (\lambda_+, \mu\lambda_{\dot{1}}),$$
$$t_- : ([\lambda_- : 1], \mu(\lambda_{\dot{1}}, \lambda_{\dot{2}})) \mapsto (\lambda_-, \mu\lambda_{\dot{2}})$$

on the overlap $\mathcal{O}(-1)|_{U_+ \cap U_-}$ we compute the transition function f_{+-} using $\Psi_{+-} = t_+ \circ t_-$ with

$$\Psi_{+-}(\lambda_-, \mu\lambda_{\dot{2}}) = (\psi_{+-}(\lambda_-), f_{+-}(\lambda_-)\mu\lambda_{\dot{2}}) = (\lambda_+, f_{+-}(\lambda_-)\mu\lambda_{\dot{2}}). \tag{1.3}$$

Going through all the details we see that

$$\Psi_{+-}(\lambda_-, \mu\lambda_{\dot{2}}) = t_+ \circ t_-^{-1} \circ t_- \left([\lambda_- : 1], \mu(\lambda_{\dot{1}}, \lambda_{\dot{2}})\right),$$
$$= t_+ \left([1 : \lambda_+], \mu(\lambda_{\dot{1}}, \lambda_{\dot{2}})\right),$$
$$= (\lambda_+, \mu\lambda_{\dot{1}}),$$

where we read off that

$$f_{+-}(\lambda_-) = \frac{\lambda_{\dot{1}}}{\lambda_{\dot{2}}} = \lambda_- = \lambda_+^{-1}. \tag{1.4}$$

The fact that $f_{+-}(\lambda_-) = \lambda_+^{-1}$ is one of the reasons that we call the tautological bundle $\mathcal{O}(-1)$.

The hyperplane bundle is another line bundle over $\mathbb{P}^1$ which is the formal dual of the tautological bundle we call the hyperplane bundle $\mathcal{O}(1)$ and say that $\mathcal{O}(1) = \mathcal{O}(-1)^*$. It will be the bundle with transition functions

$$f_{+-}(\lambda_-) = \frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}} = \lambda_+. \tag{1.5}$$

We call the tensor power bundles

$$\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}, \quad \mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m} \tag{1.6}$$

for $m \in \mathbb{N}$. The will have transition functions

$$f_{+-}(\lambda_-) = \left(\frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}}\right)^m = \lambda_+^m \quad \forall m \in \mathbb{Z}. \tag{1.7}$$

As for the global sections of these bundles, on the overlap $U_+ \cap U_-$ they should obey

$$s_+(\lambda_+) = f_{+-}(\lambda_-)s_-(\lambda_-) \tag{1.8}$$

since $f_{+-}(\lambda_-) = \lambda_+^m$ we have that

$$s_+(\lambda_+)\lambda_{\dot{1}}^m = s_-(\lambda_-)\lambda_{\dot{2}}^m. \tag{1.9}$$

This allows us to interpret the sections as

$$F(\lambda_{\dot{\alpha}}) = \begin{cases} s_+(\lambda_+)\lambda_{\dot{1}}^m & \text{on } U_+, \\ s_-(\lambda_-)\lambda_{\dot{2}}^m & \text{on } U_2, \end{cases} \tag{1.10}$$

these are homogeneous function of degree m. Now when $m < 0$ this would mean that the sections need to have a pole somewhere so they cannot be global. Thus we can see that $\mathcal{O}(m)$ only has global sections for $m \geq 0$.

Example 1.2. We have different reality conditions depending on the metric we want to work with. These reality conditions will determine how the $\sigma_{\alpha\dot{\alpha}}^\mu$ are related to the Pauli matrices, σ_i, and the identity. The three cases are

1. Euclidean $\mathbb{E}$ we take

$$\bar{x} = -\sigma_2 x \sigma_2^T. \tag{1.11}$$

2. Split signature of Kleinian, $\mathbb{K}$, take

$$\bar{x} = x. \tag{1.12}$$

3. Lorentzian $\mathbb{M}$.

$$\bar{x} = -x^T. \tag{1.13}$$

It is important to note that imposing a given reality condition will tell us how we can express the $(\sigma^\mu)_{\alpha\dot{\alpha}}$ in terms of $\mathbb{I}_2$ and the three Pauli matrices.

2 Lecture 1

Exercise 2.1. Show that if $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ then

$$x \mapsto g_1 x g_2 \tag{2.1}$$

where $g_i, g_2 \in SL(2, \mathbb{C})$ and $x = x^\mu \sigma_\mu^{\alpha\dot\alpha}$.

This is establishing the relationship between $SO(4, \mathbb{C})$ and $SL(2, \mathbb{C})$ as being

$$SO(4, \mathbb{C}) = \frac{SL(2, \mathbb{C}) \times SL(2, \mathbb{C})}{\mathbb{Z}_2}. \tag{2.2}$$

Doing this explicitly requires a lot of work using identities for the matrices $(\sigma^\mu)_{\alpha, \dot\alpha}$ and I won't give the details here. A version of this for the specific case of $\mathbb{M}$ is given in [3]

Exercise 2.2. What groups do $g_{1,2}$ belong to in all three cases, $\mathbb{E}$, $\mathbb{K}$, $\mathbb{M}$?

There are three cases to consider:

1. We start by considering the split signature case where the reality condition is given by Equation (1.12). If $x' = g_1 x g_2$ then we have that

$$\bar{g}_1 \bar{x} \bar{g}_2 = \bar{x}', \tag{2.3}$$
$$= g_1 \bar{x} g_2, \tag{2.4}$$

   from which we see that $g_i = \bar{g}_i$ and thus $g_1, g_2 \in SL(2, \mathbb{R})$. In this case the connected component of the Lorentz group will

$$SO_0(2, 2) = \frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R})}{\mathbb{Z}_2}. \tag{2.5}$$

   Note that we could have imposed an alternative reality condition which would have resulted in $g_{1,2} \in SU(1, 1)$ but since $SU(1, 1) \simeq SL(2, \mathbb{R})$ this would be equivalent.

2. For Lorentzian signature we use the reality condition given by Equation (1.13). This means that if $x' = g_1 x g_2$ we have that

$$-\bar{g}_1 x^T \bar{g}_2 = \left(g_2^T x^T g_1^T \right), \tag{2.6}$$

   which means that $g_2 = g_1^\dagger$ and as such the two copies of $SL(2, \mathbb{C})$ are not independent and we have that

$$SO_0(1, 3) = \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2}. \tag{2.7}$$

3. The Euclidean case is actually the most involved of the three with an analysis of the components of the matrices needed to see that they lie in $SU(2)$. The details are given in [1]

Exercise 2.3. For M a four dimensional spin manifold show that the decomposition of the tangent space

$$TM \simeq S \otimes \tilde{S} \tag{2.8}$$

amounts to picking a conformal structure on M.

This is discussed as Remark 1.1 in [1] but I will sketch the argument here. Before sketching a solution we first mention that choosing a conformal structure means picking an equivalence class of metrics, $[g]$, of sections of $T^*M \odot T^*M$ such that $g \sim g\prime$ if $g\prime = fg$ for f a nowhere vanishing, smooth, positive function. Now the decomposition

$$TM \simeq S \otimes \tilde{S} \tag{2.9}$$

means that we have a line sub bundle

$$L = \Lambda^2 S \otimes \Lambda^2 \tilde{S} \subset T^*M \odot T^*M. \tag{2.10}$$

This means that given symplectic forms ϵ on S and $\tilde{\epsilon}$ on $\tilde{S}$ the sections of L will be of the form $f\epsilon\tilde{\epsilon}$ however this is nothing but a metric up to a smooth positive definite function f which is $[g]$.

Exercise 2.4. Show that $\mathcal{O}(-m)$ has no global sections $\forall m \in \mathbb{N}$.

This is seen by looking at the definition of a degree $-m$ homogeneous function, that

$$F(\mu\lambda_{\dot{\alpha}}) = \mu^{-m} F(\lambda_{\dot{\alpha}}), \tag{2.11}$$

and seeing that the negative power of μ implies that F must have a pole and is hence not global.

Exercise 2.5. Check the measure $d\lambda_{\dot{\alpha}}\lambda^{\dot{\alpha}}$ in local coordinates, $\lambda_{\pm}$ on $U_{\pm}$.

To see this we just expand out the indices as follows

$$
\begin{aligned}
d\lambda_{\dot{\alpha}}\lambda^{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}(d\lambda^{\dot{\beta}})\lambda^{\dot{\alpha}}, \\
&= d\lambda^{\dot{2}}\lambda^{\dot{1}} - d\lambda^{\dot{1}}\lambda^{\dot{2}}, \\
&= \left(\lambda^{\dot{1}}\right)^2 d\lambda_+, \\
&= -\left(\lambda^{\dot{2}}\right)^2 d\lambda_-.
\end{aligned}
$$

3 Lecture 2

Exercise 3.1. Check that $\delta_q \circ \delta_{q-1} = 0$.

This is a standard exercise that involves seeing that we have lots of terms like $r^{a_0...\hat{a}_i...a_n}_{a_0...a_n} r^{a_0...\hat{a}_j...a_n}_{a_0...a_n} f_{a_0...\hat{a}_i...\hat{a}_j...a_n}$ appearing with the restrictions the other way around and the sign flipped. It might be instructive to include a simple example of, e.g the $q = 1$ case as that is

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what we need for $\mathbb{P}^1$ as the cover only has a double intersection. Consider $f_{i_0,\dots,i_p} : U_{i_0} \cap \dots \cap U_{i_p}$ then the definition of the co-boundary map, δ is that

$$\delta_p(f_{i_0,\dots,i_p}) = (p+1)r_{[i_{p+1}}f_{i_0,\dots,i_p]}, \tag{3.1}$$

where $r_{i_{p+1}}$ is short hand for the restriction map from $U_{i_0} \cap \dots \cap U_{i_p}$ to $U_{i_0} \cap \dots \cap U_{i_p} \cap U_{i_{p+1}}$. If we take the case p=1 then we can see that

$$
\begin{aligned}
\delta_1 \circ \delta_0(f_{i_0}) &= \delta_1(r_{[i_1}f_{i_0]}), \\
&= 3r_{[i_2}r_{i_1}f_{i_0]}, \\
&= \frac{1}{2}\left(r_{i_2}r_{i_1}f_{i_0} - r_{i_2}r_{i_0}f_{i_1} + r_{i_1}r_{i_0}f_{i_2} - r_{i_1}r_{i_2}f_{i_0} + r_{i_0}r_{i_2}f_{i_1} - r_{i_0}r_{i_1}f_{i_2}\right), \\
&= 0.
\end{aligned}
$$

Exercise 3.2. Check that $H^1(\mathcal{U}, \mathcal{O}(m)) = 0$ when $m \geq -1$.

This follows from what we saw in the lectures for $m < -1$ and the fact that the middle term vanishes if $m \geq -1$ and we are just left with $f_{+-} = r_{+-}^+ h_+ + r_{+-}^- h_-$ where $h_\pm$ is a holomorphic defined on all of $U_\pm$.

Exercise 3.3. Compute the Radon-Penrose transform of

$$f_{+-} = \frac{1}{z^1 z^2}. \tag{3.2}$$

This is a nice question which lets us brush the dust off of our complex analysis knowledge. We will work on the patch U_+ using coordinates $\lambda_+ = \frac{\lambda_{\dot{2}}}{\lambda_{\dot{1}}}$, In this case we know from Exercise 2.5 that the measure is

$$d\lambda_{\dot{\alpha}}\lambda^{\dot{\alpha}} = (\lambda_{\dot{1}})^2 d\lambda_+. \tag{3.3}$$

Putting this into the expression for $\phi(x)$ given by the Penrose transform we have that

$$
\begin{aligned}
\phi(x) &= -\frac{1}{2\pi i} \oint_\gamma \frac{(\lambda_{\dot{1}})^2 d\lambda_+}{z^1 z^2}, \\
&= -\frac{1}{2\pi i} \oint_\gamma \frac{d\lambda_+}{\left(x^{1\dot{1}} + x^{1\dot{2}}\lambda_+\right)\left(x^{2\dot{1}} + x^{2\dot{2}}\lambda_+\right)},
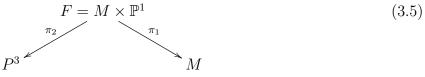\end{aligned}
$$

where γ is a curve in $U_+ \cap U_-$ and we have used the incidence relation that $z^\alpha = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}$. This integral will now pick up a contribution from the poles depending on which of them are inside the curve γ. This comes down to picking which Green's function we are interested in. If we want the usual Green's function then we pick γ so that it contains one of the poles but not the other this means that we will only pick up one residue. Say we pick γ such that $p_1 = -\frac{x^{1\dot{1}}}{x^{1\dot{2}}}$ is inside γ then we find that

$$\phi(x) = -\lim_{\lambda_+ \to p_1} \frac{1}{x^{1\dot{2}}x^{2\dot{2}}\left(\lambda_+ + \frac{x^{2\dot{1}}}{x^{2\dot{2}}}\right)} = \frac{1}{\det x}. \tag{3.4}$$

If we picked both poles to be contained in γ we would find that $\phi(x) = 0$. This problem is discussed on pages 352 and 353 of [4].

Exercise 3.4. By imposing translational invariance in one direction we reduce our space time from $\mathbb{C}^4$ to $\mathbb{C}^3$, we also know that instantons on $\mathbb{C}^4$ which are translationally invariant in one direction are monopoles on $\mathbb{C}^3$. By following through the consequences of this reduction find the monopole twistor space, P^2.

We should find that $P^2 = \mathcal{O}_{\mathbb{P}^1}(2)$. The starting point is the twistor correspondence that we saw in the lectures

$$F = M \times \mathbb{P}^1 \tag{3.5}$$

$$\pi_2 \swarrow \qquad \searrow \pi_1$$

$$P^3 \qquad\qquad\qquad M$$

where $P^3 = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. π_1 is a $\mathbb{P}^1$ fibraton due to the projectivisation of $\tilde{S}^*$ which has as homogeneous fibre coordinates the $\lambda_{\dot\alpha}$. To understand π_2 we introduce the vector fields

$$V_\alpha = \lambda^{\dot\alpha} \partial_{\alpha\dot\alpha}, \tag{3.6}$$

which form an integrable distribution, $< V_\alpha >= \mathcal{D}$, which leads to a foliation of F with 2-dimensional leaves. The twistor space P^3 is then $F/\mathcal{D}$ and has coordinates $(z^\alpha, \lambda_{\dot\alpha}) = (x^{\alpha\dot\beta}\lambda_{\dot\beta}, \lambda_{\dot\alpha})$ where the relation $x^{\alpha\dot\beta}\lambda_{\dot\beta} = z^\alpha$ is known as the Penrose incidence relation.

Before doing the dimensional reduction it is important to note that there is not a distinction between chiral and anti-chiral spinors in three dimensions and the splitting of the tangent bundle will be

$$T\mathbb{C}^3 = S \odot S, \tag{3.7}$$

This means that we will not have dotted and undotted indices but just $\alpha, \beta = 1, 2$. This means that the coordinates on $\mathbb{C}^3$ will be

$$x^{\alpha\beta} = x^{(\alpha\beta)} + x^{[\alpha\beta]} = Y^{\alpha\beta} + \varepsilon^{\alpha\beta} x^4. \tag{3.8}$$

It is this x^4 that we will impose translational invariance with respect to. The first thing this does is to make the spinorial deribvatives symmetric in their two indices, that is

$$\partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}} = \frac{\partial}{\partial x^{\beta\alpha}} = \partial_{\beta\alpha}. \tag{3.9}$$

We can still construct vectors which generate null translation, $V_\beta = \lambda^\alpha \partial_{(\alpha\beta)}$ and these vectors still generate an integrable distribution which results in a foliation of F with two dimensional leaves. The difference comes when we consider the kernel of these vectors. In the four dimensional case we had that

$$V_\alpha(z^\beta) = 0 \tag{3.10}$$

but now we would find that

$$V_\gamma(x^{\alpha\beta}\lambda_\beta) = \lambda^\delta \lambda_\beta \partial_{(\delta\gamma)} x^{\alpha\beta} = \lambda^\delta \lambda_\beta \frac{1}{2} \left(\delta_\delta^\alpha \delta_\gamma^\beta + \delta_\delta^\beta \delta_\gamma^\alpha \right) = \frac{1}{2} \lambda^\alpha \lambda_\gamma, \tag{3.11}$$

so our incidence relation will not be $z^\alpha = x^{\alpha\beta}\lambda_\beta$. We take in fact the incidence relation $z = x^{\alpha\beta}\lambda_\alpha\lambda_\beta$, which means that z is of weight 2, as this satisfies

$$V_\gamma(z) = V_\gamma(x^{\alpha\beta}\lambda_\alpha\lambda_\beta) = \lambda^\delta \lambda_\alpha \lambda_\beta \partial_{(\delta\gamma)} x^{\alpha\beta} = \lambda^\delta \lambda_\alpha \lambda_\beta \frac{1}{2} \left(\delta_\delta^\alpha \delta_\gamma^\beta + \delta_\delta^\beta \delta_\gamma^\alpha \right) = 0, \tag{3.12}$$

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so they do live in the kernel of the distribution. This tells us that P^2 has coordinates (z, λ_α) where the λ_α are homogeneous on a $\mathbb{P}^1$ and the $z's$ are of weight two, this tells us that the three dimensional twistor space will be the bundle

$$P^2 = \mathcal{O}_{\mathbb{P}^1}(2), \tag{3.13}$$

which is what we wanted.

It would be interesting to check some of the details about getting a solution to the monopole equations, $f_{\alpha\beta} = \nabla_{\alpha\beta}\phi$, by putting a holomorphic bundle on P^2. Also maybe try and see how this is related to the self-dual string twistor space.

Exercise 3.5. Compute the charge of the BPST instanton,. The following integral will be useful

$$\int d^4 x \frac{1}{(x^2 + \Lambda^2)^4} = \frac{\pi^2}{6\Lambda^4}. \tag{3.14}$$

I have done this before, compute $tr(f_{ab}f_{ab})$, where we were given the f_{ab} in the lecture, and then integrate to get 1. The components f_{ab} are also given in Equation 3.31 of [1].

4 Lecture 3

Exercise 4.1. Use the Bianchi identity and the transversal gauge condition to obtain the recursion relations given in the notes. That is check the computations leading up to Equation 5.10 in [1].

This is a long computation that I probably won't do.

Exercise 4.2. Show that for F^- the anti self-dual piece of the two form curvature, F, the following holds

$$\int \text{tr} \left(F^- \wedge F^- \right) = -\frac{1}{2} \int \text{tr} \left(F \wedge \star F \right) + \frac{1}{2} \int \text{tr} \left(F \wedge F \right). \tag{4.1}$$

This is a fairly straight forward exercise and just uses that the antiself-dual part of F is given by

$$F^- = \frac{1}{2} \left(F - \star F \right), \tag{4.2}$$

plug this into the left hand side of Equation (4.1) and you will get the result.

5 Lecture 4

These exercises are related to content from [5, 6]

Exercise 5.1. Show that the twistor space for higher gauge theory can be interpreted as

$$P = T^* \mathbb{P}^3 \otimes \mathcal{O}_{\mathbb{P}^3}(2). \tag{5.1}$$

Hint the one-forms will look like

$$d\lambda_A \lambda_B \epsilon^{ABCD} \omega_{CD} \tag{5.2}$$

where ω_{CD} can be shown to correspond to x^{CD}.

6   Lecture 5

These exercises are related to the content of [7, 8]

Exercise 6.1. Check that $\tilde\tau$ is killed by contracting with vectors in $\mathcal{D}$, in other words show that

$$i_\mathcal{D}\tilde\tau = 0. \tag{6.1}$$

We have from the lectures that

$$\tilde\tau = \lambda^{\dot\alpha}\nabla\lambda_{\dot\alpha} = \lambda^{\dot\alpha}\left[d\lambda_{\dot\alpha} - \omega_{\dot\alpha}{}^{\dot\beta}\lambda_{\dot\beta}\right] \tag{6.2}$$

and $\mathcal{D}$ is the distribution generated by the vector $V_B = \lambda^{\dot\beta}E_{B\dot\beta} + \lambda^{\dot\alpha}\lambda_{\dot\gamma}\omega_{B\dot\alpha\dot\beta}{}^{\dot\gamma}\frac{\partial}{\partial\lambda_{\dot\beta}}$. Here the $E^{A\dot\alpha}$ form a vielbein for the curved four manifold M, the $E_{A\dot\alpha}$ are their dual vectors and $\omega_{B\dot\alpha\dot\beta}{}^{\dot\gamma}E^{B\dot\alpha}$ are the connection one forms for the spin connection on $\tilde S$. This means that the $E^{A\dot\alpha}$ should be independent of the $\lambda_{\dot\alpha}$ just as the $\lambda_{\dot\alpha}$ are independent of the spacetime coordinates. e.g.

$$\frac{\partial}{\partial\lambda_{\dot\beta}}E^{B\dot\alpha} = 0, \quad E_{A\dot\gamma}\lambda_{\dot\beta} = 0. \tag{6.3}$$

With this we can compute

$$
\begin{aligned}
i_{V_A}\tilde\tau &= \lambda^{\dot\alpha}V_A(\lambda_{\dot\alpha}) - \lambda^{\dot\alpha}i_{V_A}(\omega_{\dot\alpha}{}^{\dot\beta})\lambda_{\dot\beta}, \\
&= \lambda^{\dot\alpha}\lambda^{\dot\beta}\lambda_{\dot\gamma}\omega_{A\dot\beta\dot\delta}{}^{\dot\gamma}\frac{\partial}{\partial\lambda_{\dot\delta}}\lambda_{\dot\alpha} - \lambda^{\dot\alpha}\lambda_{\dot\beta}\lambda^{\dot\gamma}\omega_{B\dot\delta\dot\alpha}{}^{\dot\beta}E_{A\dot\gamma}(E^{B\dot\delta}), \\
&= \lambda^{\dot\alpha}\lambda^{\dot\beta}\lambda_{\dot\gamma}\omega_{A\dot\beta\dot\alpha}{}^{\dot\gamma} - \lambda^{\dot\alpha}\lambda_{\dot\beta}\lambda^{\dot\gamma}\omega_{A\dot\gamma\dot\alpha}{}^{\dot\beta}, \\
&= 0.
\end{aligned}
$$

Now we just note that the integral contraction is both $+$ and $C^\infty(M)$ linear so if this is true for the basis V_A it will be true for any vector in the distribution $\mathcal{D}$ generated by V_A giving us the desired result.

Exercise 6.2. Compute the explicit for of $d\tilde\tau$ from $\tilde\tau = \lambda^{\dot\alpha}\nabla\lambda_{\dot\alpha}$.

This is another computation which uses the explicit form of $\tilde\tau$ as well as the relationship between the spin connection and the curvature. The first thing to note is that since the index is raised we will have

$$\nabla\lambda^{\dot\alpha} = d\lambda^{\dot\alpha} + \omega^{\dot\alpha}{}_{\dot\beta}\lambda^{\dot\beta}. \tag{6.4}$$

Next note that

$$d\tilde\tau = d\lambda^{\dot\alpha}\wedge\nabla\lambda_{\dot\alpha} - \lambda^{\dot\alpha}\lambda_{\dot\beta}d\omega_{\dot\alpha}{}^{\dot\beta} - \lambda^{\dot\alpha}\omega_{\dot\alpha}{}^{\dot\beta}\wedge d\lambda_{\dot\beta}. \tag{6.5}$$

Now we can compute that

$$\nabla\lambda^{\dot\alpha}\wedge\nabla\lambda_{\dot\alpha} = d\lambda^{\dot\alpha}\wedge\nabla\lambda_{\dot\alpha} + \omega^{\dot\alpha}{}_{\dot\gamma}\wedge\lambda^{\dot\gamma}d\lambda_{\dot\alpha} - \omega_{\dot\gamma}{}^{\dot\alpha}\wedge\omega_{\dot\alpha}{}^{\dot\beta}\lambda^{\dot\gamma}\lambda_{\dot\beta}, \tag{6.6}$$

so using the antisymmetry of the spin connection we have that

$$d\tilde\tau = \nabla\lambda^{\dot\alpha}\wedge\nabla\lambda_{\dot\alpha} - \lambda^{\dot\alpha}\lambda_{\dot\beta}d\omega_{\dot\alpha}{}^{\dot\beta} + \omega_{\dot\gamma}{}^{\dot\alpha}\wedge\omega_{\dot\alpha}{}^{\dot\beta}\lambda^{\dot\gamma}\lambda_{\dot\beta}. \tag{6.7}$$

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Note that the connection one forms will satisfy

$$\Omega_{\dot{\alpha}}{}^{\dot{\beta}} = d\omega_{\dot{\alpha}}{}^{\dot{\beta}} - \omega_{\dot{\alpha}}{}^{\dot{\gamma}} \wedge \omega_{\dot{\gamma}}{}^{\dot{\beta}}, \tag{6.8}$$

where $\Omega_{\dot{\alpha}}{}^{\dot{\beta}}$ is the curvature two form. Using this identity we have that

$$d\tilde{\tau} = \nabla\lambda^{\dot{\alpha}} \wedge \nabla\lambda_{\dot{\alpha}} - \lambda^{\dot{\alpha}}\lambda_{\dot{\beta}}\Omega_{\dot{\alpha}}{}^{\dot{\beta}}. \tag{6.9}$$

If we knew more about the connection two form and its relation to the Riemann curvature tensor we could do more with this expression.

References

[1] M. Wolf. A First Course on Twistors, Integrability and Gluon Scattering Amplitudes, J. Phys. A **43** (2010) 393001. [arXiv:1001.3871 [hep-th]].

[2] S. A. Huggett and K. P. Todd. An Introduction to Twistor Theory, Second Edition. LMS Student Texts 4, Cambridge University Press, 1994.

[3] H. Müller-Kirsten and A. Wiedemann. Introduction to Supersymmetry, Second Edition. World Scientific Lecture Notes in Physics Vol. 80, World Scientific, 2010.

[4] R. S. Ward and R. O. Wells. Twistor Geometry and Field Theory. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1990.

[5] C. Saemann and M. Wolf. On Twistors and Conformal Field Theories from Six Dimensions, J. Math. Phys. **54** (2013) 013507. [arXiv:1111.2539 [hep-th]].

[6] C. Saemann and M. Wolf. Non-Abelian Tensor Multiplet Equations from Twistor Space, Commun. Math. Phys. **328** (2014) 527. [arXiv:1205.3108 [hep-th]].

[7] M. Wolf. Self-Dual Supergravity and Twistor Theory, Class. Quant. Grav. **24** (2007) 6287. [arXiv:0705.1422 [hep-th]].

[8] M. Wolf. Contact Manifolds, Contact Instantons, and Twistor Geometry, JHEP **1207** (2012) 074. [arXiv:1203.3423 [hep-th]].