Topology in Physics
Some recent applications: 2

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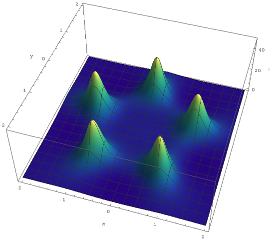


Solitons in chiral magnets

Outline

Solitons in real materials

- In the previous lecture we saw several examples of mathematical models with soliton solutions.
- Now we want to see some examples of applications of solitons.
- The theoretical models of magnetic skyrmions originate in the work of Bogdanov and collaborators starting in 1989.

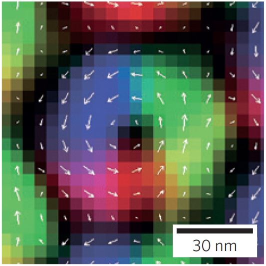


Figure 1: Experimental image of a magnetic skyrmion from Nagaosa and Tokura 2013.

$O(3)$ sigma model

On Monday we met the $O(3)$ sigma model.

- The static energy is

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x\,(\nabla m)^2\,, \qquad m : \mathbb{R}^2 \to S^2$$

- Finite energy solutions extend to $m : S^2 \to S^2$ with topological charge

$$Q[m] = \frac{1}{4\pi}\int d^2x\,(m \cdot \partial_1 m \times \partial_2 m)\,.$$

- There is a Bogomol'nyi bound

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_1 m \pm m \times \partial_2 m)^2 + 2\pi|Q[m]|.$$

O(3) Bogomol'nyi equations

- The minimum energy configurations solve the Bogomol'nyi equations:

$$\partial_1 m \pm m \times \partial_2 m = 0.$$

- These are much easier to study using complex stereographic coordinates

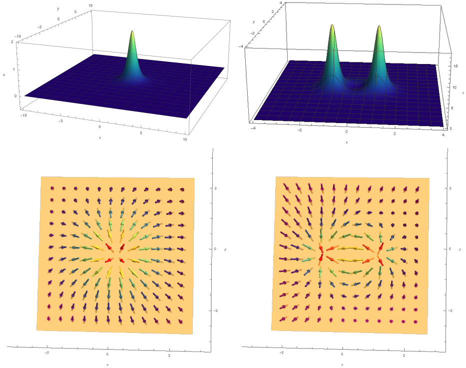
$$m_1 + i m_2 = \frac{2w}{1+|w|^2}, \quad m_3 = \frac{1-|w|^2}{1+|w|^2}.$$

The Bogomol'nyi equations, for local $\mathbb{C}$ coord $z = x + iy$, are

$$\partial_z w = 0 \quad \text{or} \quad \partial_{\bar{z}} w = 0.$$

- These are solved by rational maps $w(z) = \frac{z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$.

Examples of lumps



Plots of energy density and m for $w = z$ and $w = (z-1)(z+1)$.

Real materials

- To describe real materials we need to add extra terms to the energy functional.
- For applications to nuclear matter the 4th order Skyrme term and the 6th order sextic term.
- For magnetic matter a first order term is needed to account for spin orbit interaction (anti-symmetric exchange)

$$E[m] = \int_{\mathbb{R}^2} d^2x \left[\frac{1}{2} (\nabla m)^2 + \kappa m \cdot \nabla_{-\alpha} \times m + U(m_3) \right]$$

DM Interaction

Dzyaloshinskii (1958) and Moriya (1960) realised that atomic spin orbit effects lead to a contribution of the form

$$m \cdot \nabla_{-\alpha} \times m = m \cdot \sum_{i=1}^{2} e_i^{-\alpha} \times \partial_i m.$$

$e_i^{-\alpha} = R_3(-\alpha) e_i$ are rotations of the standard basis vectors.

The symmetry of the energy functional is the diagonal subgroup of $SO(2) \times SO(2)$. (Translations are also symmetries)

This most commonly studied DM terms are the $\alpha = 0$ "Bloch" type and the $\alpha = \frac{\pi}{2}$ "Néel" type.

Boundary term

- From an analysis point of view the DM term can cause issues with the variational problem for $E[m]$.
- This is because varying it leads to the DM term leads to the boundary term
$$BT[m] = -\kappa \int_{\mathbb{R}^2} d^2x \, \nabla \cdot (m \times \delta m)$$
- If the field falls off as $\frac{1}{r}$, like the $O(3)$ lumps, then this term is not set to zero by $\lim_{\vec{x}\to\infty} m \to m_\infty \in \mathcal{V}$, $\lim_{\vec{x}\to\infty} \delta m \to 0$.
- One solution is to subtract this boundary term. Here we want to showcase the solutions so will not subtract it.

Potential $V(m_3)$ I

The potential has the form

$$V(m_3) = B(1 - m_3) - A\left(1 - m_3^2\right)$$

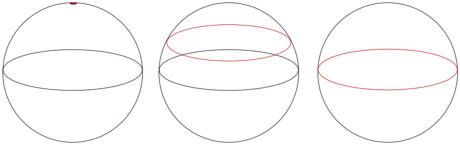
First term is a Zeeman term, minimised by m_3 pointing in the positive z direction everywhere. Second term is anisotropy term, depends on the sign of A.

- $A < 0$, easy axis potential. This prefers spins to all point in the $+z$ or - z direction.
- $A > 0$, easy plane potential. Prefers spins to lie in the $x - y$ plane.

Potential $V(m_3)$ II

The ground state depends on the relative sizes of A and B. Can assume $B > 0$ w.l.o.g

- $B \geq 2A$ the minimum is $V(m_3) = 0$ at $m_3 = 1$
- $B < 2A$ the minimum is $V(m_3) = B - A - \frac{B^2}{4A}$ at $m_3 = \frac{B}{2A}$.
- The vacuum manifolds are



The Solvable line $B = 2A$

- The extended $O(3)$ model has exact solutions for the easy plane case with $A = \frac{B}{2}$.
- The potential can be written as

$$V(m_3) = \frac{B}{2}(1 - m_3)^2$$

- and the energy functional is

$$E[m] = \int_{\mathbb{R}^2} d^2x \left[\frac{1}{2}(\nabla m)^2 + \kappa m \cdot (\nabla_{-\alpha} \times m) + \frac{B}{2}(1 - m_3)^2 \right]$$

Axially-symmetric configurations I

- A particularly nice class of field configurations, respecting the $SO(2)$ symmetries of the model are the hedgehog fields

$$m = \begin{pmatrix} \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix}$$

  with $\Theta = \Theta(r)$ and $\Phi = n\phi + \gamma$.
- They have topological charge $Q[m] = -n$.
- Hedgehogs respect the full $O(2)$ symmetry if $\gamma = \frac{\pi}{2} - \alpha$.
- Configurations with $n > 1$ are unstable!

Axially-symmetric configurations II

- Searching for solutions of the hedgehog form the equation of motion becomes

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin(2\Theta)}{2r^2} - 2\kappa\frac{\sin^2\Theta}{r} + B\sin\Theta\,(1-\cos\Theta)\,.$$

- For Hedgehog configurations the $O(3)$ Bogomol'nyi equations are

$$\frac{d\Theta}{dr} + \frac{\sin\Theta}{r} = 0$$

solved by

$$\Theta(r) = 2\arctan\left(\frac{2}{\lambda r}\right) \qquad \lambda \in \mathbb{R}.$$

- The EOM for the $O(3)$ model is equivalent to half of the above EOM:

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin(2\Theta)}{2r^2}$$

Axially-symmetric configurations III

- For $\Theta(r)$ to satisfy

$$2\kappa\frac{\sin^2\Theta}{r} = B\sin\Theta\left(1-\cos\Theta\right).$$

$\Rightarrow$

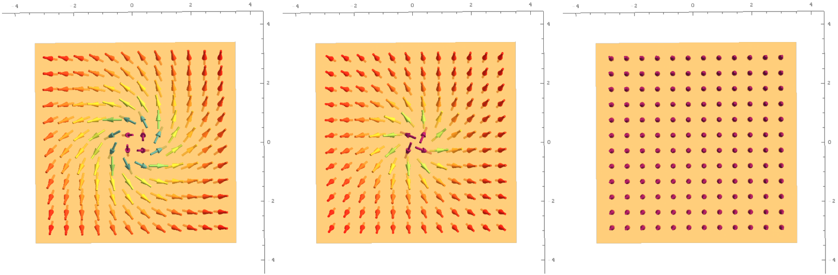
$$\lambda = \frac{B}{\kappa}$$

- Thus there are $Q = -1$ hedgehog configurations with chiral magnets on the solvable line $B = 2A$.
- They have energy

$$E[m] = 4\pi$$

Axially-symmetric configurations IV

- Unlike in most theories with topological solitons there is a finite energy barrier between the skyrmion and the vacuum.
- To see this consider hedgehog configurations with $\gamma \neq \frac{\pi}{2} - \alpha$. These are not solutions of the EOM but still have energy 4π.
- Examples with $\alpha = 0$ and $\gamma = \frac{\pi}{4}, \gamma = \frac{\pi}{8}, \gamma = 0$.



Critically coupled model I

- By tuning the coupling of the DM term and the potential to $B = \kappa^2$ we can find a whole family of mutli skyrmion configurations.
- This critically coupled model can be interpreted as a gauged version of the $O(3)$ model.
- The connection and curvature are

$$A_i = -\kappa e_i^{-\alpha}, \qquad F_{12} = \kappa^2 e_3,$$

- The Covariant derivative is

$$D_i m = \partial_i m - \kappa e_i^{-\alpha} m, \qquad e_i^{-\alpha} = R(-\alpha) e_i.$$

- A quick computation gives

$$\frac{1}{2}\left[(D_1 m)^2 + (D_2 m)^2\right] = \frac{1}{2}(\nabla m)^2 + \kappa m \cdot (\nabla_{-\alpha} \times m) + \frac{\kappa^2}{2}\left(1 + m_3^2\right)$$

Critically coupled model II

- In terms of the covariant derivative the energy functional is

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x \left[(D_1 m)^2 + (D_2 m)^2 - \kappa^2 m_3\right]$$

- A useful identity re-expresses the covariant derivative in terms of the topological charge density as

$$\frac{1}{2}\left[(D_1 m)^2 + (D_2 m)^2\right] = \frac{1}{2}\left(D_1 m + m \times D_2 m\right)^2 + \kappa^2 m_3 \\ + m \cdot \partial_1 m \times \partial_2 m + \kappa\left(\partial_1 m_2^\alpha - \partial_2 m_1^\alpha\right)$$

- This leads to a bound for the energy

$$E[m] \geq 4\pi\left(Q[m] + \Omega[m]\right)$$

Critically coupled model III

- There is equality when the Bogomol'nyi equations are satisfied,

$$D_1 m + m \times D_2 m = 0.$$

- The quantities in the bound are

$$Q[m] = \frac{1}{4\pi} \int_{\mathbb{R}^2} d^2x \left(m \cdot \partial_1 m \times \partial_2 m \right),$$
$$\Omega[m] = \frac{\kappa}{4\pi} \int_{\mathbb{R}^2} d^2x \left(\partial_1 m_2^\alpha - \partial_2 m_1^\alpha \right)$$

- These are the topological charge and the total vortex strength.

Critically coupled model IV

- This bound is different from the familiar $O(3)$ sigma model as the $Q[m]$ appears not $|Q[m]|$.
- The integrand of the total vortex strength is $e_3 \cdot (\nabla_{-\alpha} \times m)$, it is the boundary piece of the DM term.
- For some configurations the integral of the total vortex strength is not well defined and needs to be regularised on a disc with a circular boundary.
- The best way to understand the Bogomol'nyi equations is to work in stereographic coordinates

$$w = \frac{m_1 + im_2}{1 + m_3}, \qquad \text{and} \;\; v = \frac{1}{w}.$$

Complex coordinates

- In stereographic coordinates the Bogomol'nyi equations become

$$\partial_{\bar{z}} v = -\frac{i}{2}\kappa e^{i\alpha}.$$

- This has the general solution

$$v = -\frac{i}{2}e^{i\alpha}\bar{z} + f(z)$$

for an arbitrary holomorphic function f.

- When f is rational, $f(z) = \frac{P(z)}{Q(z)}$, with P, Q of degree p, q then

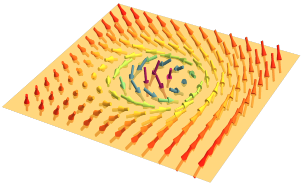
$$E[m] = 4\pi\text{max}(p, q+1)$$

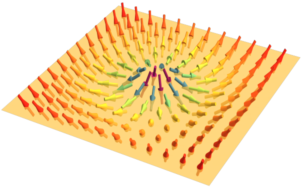
when $p = q + 1$ this is the regularised energy.

Proving this is a worthwhile computation.

A zoo of skyrmion configurations

- There are many nice examples of solutions found by picking your favourite holomorphic function.
- The simplest choice of $f(z) = 0$ leads to hedgehog Bloch and Neél skyrmions depending on if $\alpha = 0$ or $\alpha = \frac{\pi}{2}$.





$E = 4\pi$ I

In this sector there is a four dimensional family of solutions

$$v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + az + b, \quad a, b \in \mathbb{C}.$$

By translations and rotations can fix everything but $|a|$.
Changing $|a|$ corresponds to stretching or squeezing the energy density of the solution.

$E = 4\pi$ II

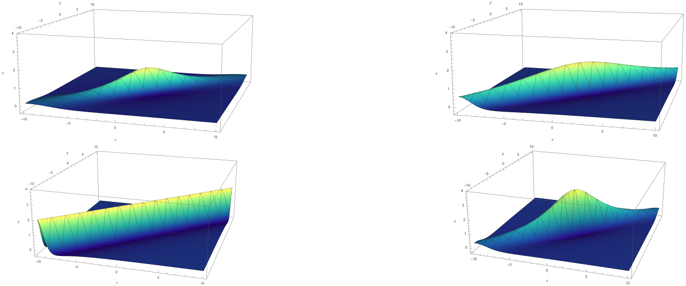


Figure 2: Stretching and squeezing for the configuration $v = -\frac{i}{2}\bar{z} + az$ with $a = 0.3$ (top left), $a = 0.4$ (top right), $a = 0.5$ (bottom left) and $a = 0.7$ (bottom right).

$E = 4\pi$ III

When $|a| > \frac{1}{2}$, $Q = 1$ and the solutions look like an anti-skyrmions.

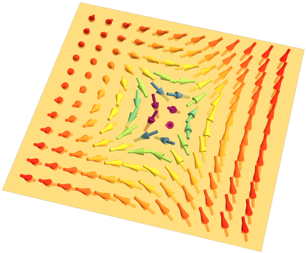


Figure 3: For $v = -\frac{i}{2}\bar{z} + 3iz$ we have an anti skyrmion with $Q[v] = 1$.

Line defect I

- Within the family of solutions with regularised energy 4π a particularly interesting type of solution is

$$v = -\frac{i}{2}e^{i\alpha}\left(\bar{z} + e^{i\delta}z\right).$$

- These solutions have a whole line where $m_3 = -1$, $\varphi = -\frac{\delta}{2} \pm \pi$.
- This is an example of a solution which does not extend to a map of spheres.

Line defect II

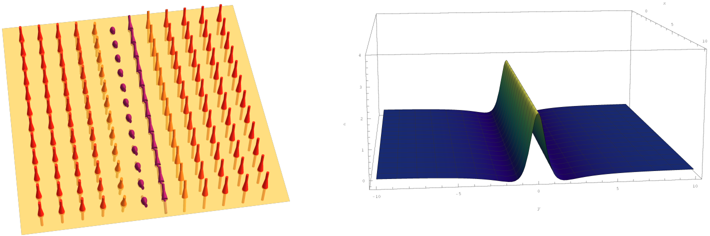


Figure 4: Left to right: magnetisation plot and energy density plot for the solution $v = -\frac{i}{2}\left(\bar{z} - z\right)$

Line defect III

- A feature of the critically coupled model is that linear solutions can pass through the line defect and change degree.
- This is one of two places where we see a line of south poles, the others are skyrmion bag configurations.
- These solutions are one of the reasons we need to work with the regularised energy.

$E = 8\pi$ I

- Moving to the next energy sector there is an eight dimensional family of solutions
$$v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + \frac{az^2 + bz + c}{dz + e},$$
with $a, b, c, d, e \in \mathbb{C}$, and $(a, b, c, d, e) \sim \lambda(a, b, c, d, e),\ \lambda \in \mathbb{C}^*$.
- In this family we can find solutions which are a combination of skyrmions and anti-skyrmions and solutions which just consist of anti-skyrmions.

$E = 8\pi$ II

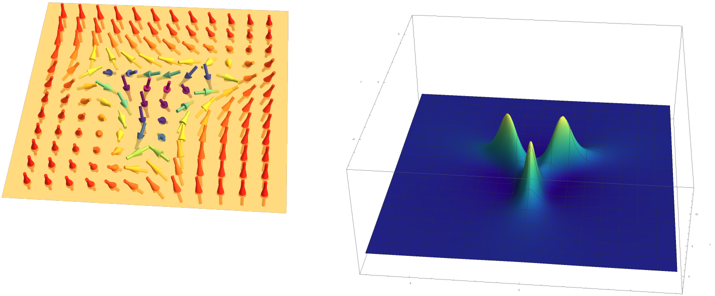


Figure 5: Magnetisation and energy density for $v = -\frac{i}{2}\bar{z} + \frac{1}{2}z^2$. This is an example of a configuration involving a skyrmion and three anti-skyrmions.

Skyrmion bags I

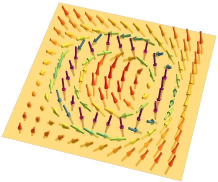
- An interesting feature that arises at $E = 8\pi$ are the $Q = 0$ skyrmion bags or sacks. These have been seen numerically in the full model by Foster and collaborators (2018) and Rybakov and Kiselev (2018).
- In the critically coupled model they arise when

$$v = -\frac{i}{2}\kappa e^{i\alpha}\left(\bar{z} - \frac{R^2}{z}\right).$$

  with $R \in \mathbb{R}_{>0}$ the radius of the bag.
- Like the basic holomorphic solution these are invariant under spin-isospin rotations.
- In the numerics there are bags with skyrmions inside them but these are not possible in the critically coupled model.
- As bags have $Q = 0$ they are non-topological solitons.

Skyrmion bags II

An example of a bag is $v = -\frac{i}{2}\left(\bar{z} - \frac{16}{z}\right)$



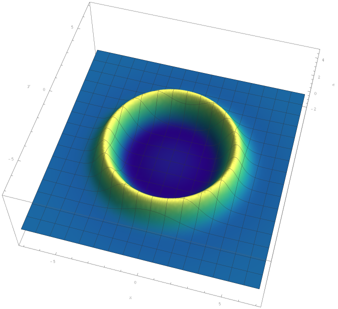


Figure 6: Magnetisation and energy density for the skyrmion bag defined by $v = -\frac{i}{2}\left(\bar{z} - \frac{16}{z}\right)$.

Higher energy

The higher energy solutions have been less studied but we can find solutions with interesting configurations arising.

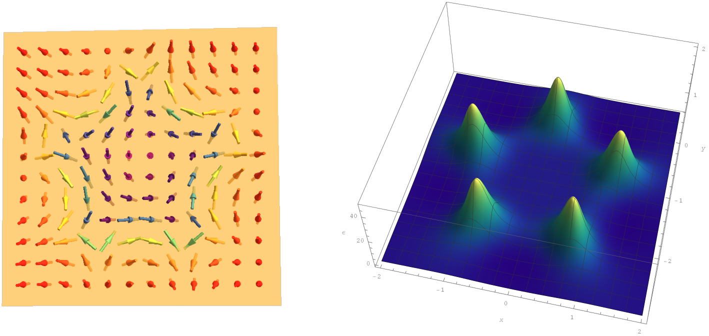


Figure 7: Magnetisation and energy density for $v = -\frac{i}{2}\bar{z} + \frac{1}{2}z^4$. There are five anti-skyrmions surrounding one skyrmion at the centre.

Sources of more information

The critically coupled model and its generalisations are an active area of study. For more information about magnetic skyrmions check out:

- B. Barton-Singer, CR and B. J. Schroers. Magnetic skyrmions at Critical Coupling. CMP 2020.
- B. J. Schroers. Gauged Sigma Models and Magnetic Skyrmions. Sci Post Phys 2019.
- V. M. Kuchkin, B. Barton-Singer, F. N. Rybakov, S. Blügel, B. Schroers, N. S. Kiselev Magnetic skyrmions, chiral kinks and holomorphic functions, 2020.
- A. Bogdanov and A. Hubert. Thermodynamically stable magnetic vortex states in magnetic crystals. Journal of Magnetism and Magnetic Materials 1994
- N. Nagaosa and Y. Tokura. Topological properties and dynamics of magnetic skyrmions. Nature nanotechnology 2013.