Topology in Physics
Some recent applications: 1

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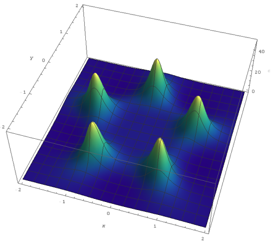


A primer on Topological Solitons

Outline

Good References

Topological Solitons show up under many guises in different areas of physics.

- Instantons, monopoles and vortices in Gauge theories.
- Vortices in superconductors and superfluids.
- Skyrmions in magnetic materials.
- Kinks or domain walls in one dimensional magnetic materials.

There are several good references if you want to know more about any of the topics that I will mention:

- Coleman: Aspects of Symmetry,
- Manton and Sutcliffe: Topological Solitons,
- Rajaraman: Solitons and Instantons,
- and many more....

A lot of this first lecture will follow the book by Coleman.

What is a soliton?

The first question to answer is what we mean by a soliton.

- In most field theories finite energy, non-singular, solutions will dissipate to the vacuum given enough time.
- There are examples of field configurations for which this is not true. This is essentially what we mean by a soliton.
- All of the examples that we construct will be static field configurations that sustain themselves through self-interaction.
- In a Lorentz or Galilean invariant theory these static solitons can be boosted to give time-dependent solutions

Definition of a soliton

There are two definitions to have in mind.

1. (weak) A finite energy, non-singular, solution of the classical equations of motion that does not dissipate and can propagate.
2. (strong) The weak definition plus survives scattering with other such solutions.

Historically the weak definition defines a lump while the strong definition defines a soliton e.g in Coleman. We will use them interchangeably.

What about a topological soliton? Same idea but stability is related to the topology of the space of field configurations.

Examples

The typical first examples of theories possessing topological solitons are in 1+1 dimensions.

- ϕ^4 theory with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{2}\left(\phi^2 - a^2\right)^2,$$

  $\lambda \in \mathbb{R}_+$ and $a^2 = \frac{\mu^2}{\lambda}$ in terms of the usual mass parameter.

- Sine-Gordon theory with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\alpha}{\beta^2}\left(1 - \cos\left(\beta\phi\right)\right),$$

  $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$.

These examples showcase a lot of the features that are encountered when studying solitons. $\mu = 0, 1$, $\mathrm{diag}(g) = (1, -1)$.

Vacuum manifolds

Before exploring these examples in more detail we want to give some generalities about $1+1D$ field theories.

- For a theory with energy

$$E = \int dx \left[\frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\partial_1 \phi)^2 + U(\phi) \right]$$

  the energy is bounded below if $U(\phi)$ is. We will assume that $E \geq 0$ if not can add constant to E such that this is true.

- A ϕ_0 such that

$$\partial_0 \phi_0 = \partial_1 \phi_0 = U(\phi_0) = 0$$

  is called a vacuum or ground state.

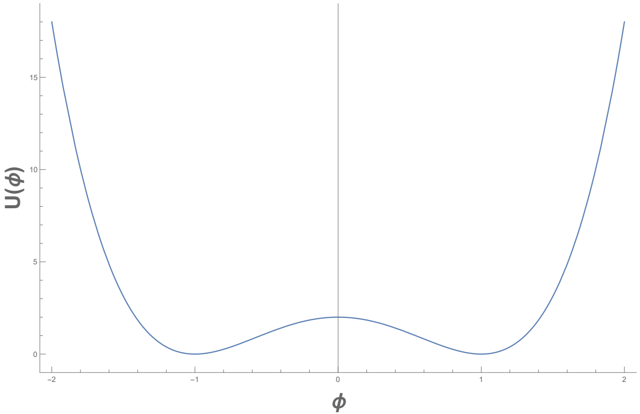
- The set of ground states,

$$\mathcal{V} = \{\phi | U(\phi) = 0\}$$

  is often called the vacuum manifold.

ϕ^4 Theory

For ϕ^4 theory the potential, $\frac{\lambda}{2}\left(\phi^2 - a^2\right)^2$ has two, repeated, zeros at $\phi = \pm a$

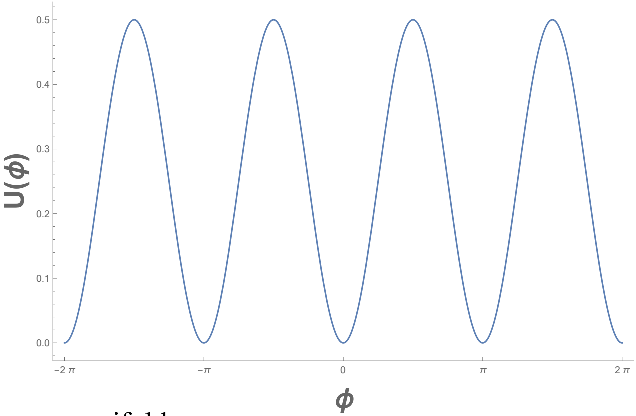


and the vacuum manifold is

$$\mathcal{V} = \{-a, a\}.$$

Sine-Gordon Theory

For Sine-Gordon theory the potential, $\frac{\alpha}{\beta^2}\left(1 - \cos\left(\beta\phi\right)\right)$ is periodic with period $\frac{2\pi}{\beta}$



with the vacuum manifold

$$\mathcal{V} = \{\phi = \frac{2\pi}{\beta}k | k \in \mathbb{Z}\}.$$

Kinks in one dimension

To find static soliton solutions we solve

$$\delta \int dx \left[\frac{1}{2}\left(\partial_1 \phi\right)^2 + U(\phi)\right] = 0$$

subject to

$$\lim_{x \to \pm\infty} \phi(x) = \phi_\pm \in \mathcal{V}.$$

This asymptotic condition is there to ensure finiteness of the energy.

- If $\mathcal{V}$ has one-point then the only solution is $\phi_0 \in \mathcal{V}$.
- If $\mathcal{V}$ has more elements then there exist solitons, called kinks in $1D$ interpolating between neighbouring elements.

Topology

The asymptotics, $(\phi_-, \phi_+) \in \mathcal{V} \times \mathcal{V}$ classify the solutions.

- $\phi_- < \phi_+$, ϕ monotonically increasing and the solution is a soliton,
- $\phi_- > \phi_+$, ϕ monotonically decreasing and the solution is an anti soliton.

This is where the topology appears.

- When $\phi_- = \phi_+$ the configuration can be deformed to ϕ_+, via a finite energy path.
- $\phi_- \neq \phi_+$ the configuration can not be deformed to one in $\mathcal{V}$.

This means that the space of field configurations is disconnected. Topology comes from the connected components of $\mathcal{V}$, $\pi_0(\mathcal{V})$.

Topology in ϕ^4

For the ϕ^4 theory $\mathcal{V} = \{-a, a\}$, $\pi_0(\mathcal{V})$ is also a two element set, and there are four possibilities for the asymptotic configuration:

1. $(-a, -a)$ equivalent to $\phi = -a$,
2. (a, a) equivalent to $\phi = a$,
3. $(-a, a)$ a soliton,
4. $(a, -a)$ an anti soliton.

We can differentiate between these by the topological charge

$$N = \frac{1}{2a} \int_{-\infty}^{\infty} dx \frac{d\phi}{dx} = \frac{\phi_+ - \phi_-}{2a} \in \mathbb{Z}.$$

For a soliton $N = 1$, and and anti soliton $N = -1$.

Topology in SG

- Some thing similar can be done in Sine-Gordon theory.
- Here $\pi_0\left(\mathcal{V}\right) = \mathbb{Z}$ so there is a richer topology as the topological charge is not limited to $-1, 0, 1$.
- Topological charge is

$$N = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dx \frac{d\phi}{dx} = \beta\frac{\phi_+ - \phi_-}{2\pi} \in \mathbb{Z}.$$

- Again kinks have $N > 0$ and anti kinks have $N < 0$.

Bogomol'nyi equations and kinks

In $1+1D$ there is a completing the square argument, explained in detail in the notes, that gives

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\partial_1 \phi \mp \sqrt{2U(\phi)} \right)^2 \pm \sqrt{2U(\phi)} \frac{d\phi}{dx} \right] \geq \left| \int_{\phi_-}^{\phi_+} d\phi \sqrt{2U(\phi)} \right|$$

As $U(\phi)$ is bounded there is a "superpotential" W such that $U = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$. Thus

$$E \geq \left| W(\phi_+) - W(\phi_-) \right|,$$

with equality when the Bogomol'nyi equations are satisfied

$$\partial_1 \phi \mp \sqrt{2U(\phi)} = 0$$

Solving Bogomol'nyi

- The Bogomol'nyi equations can be solved by quadratures

$$x - x_0 = \pm \int_{\phi_0}^{\phi} \frac{d\varphi}{\sqrt{2U(\varphi)}}.$$

- x_0 is the centre of the kink, $\phi_0 = \phi(x_0)$ is a point between ϕ_- and ϕ_+.
- We will see explicit examples for both the ϕ^4 and Sine-Gordon theories.
- The Bogomol'nyi equations imply the full equations of motion so their solutions give solitons.
- $\phi^4 \rightsquigarrow$ lumps (weak definition).
- Sine-Gordon $\rightsquigarrow$ solitons (strong definition).

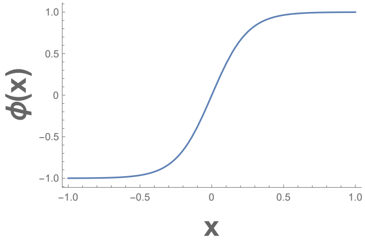
ϕ^4 kinks

For the kink in ϕ^4 theory we take $\phi_\pm = \pm a$ and $\phi_0 = \phi(0) = 0$ such that

$$x = \pm\frac{1}{\mu}\int_0^\phi \frac{d\varphi}{\frac{\varphi^2}{a^2} - 1} = \mp\frac{1}{\mu}\,\mathrm{arctanh}\left(\frac{\phi}{a}\right).$$

This inverts to give the kink

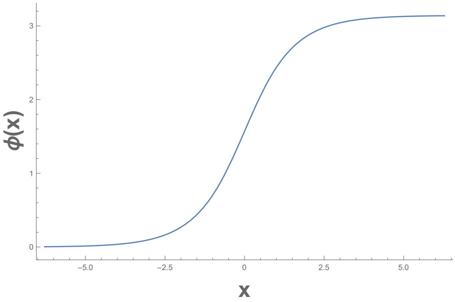
$$\phi(x) = a\tanh\left(\mu x\right).$$



SG kinks

A similar approach works in Sine-Gordon theory to give

$$\phi(x) = \frac{4}{\beta} \arctan\left(\exp\left(\sqrt{\alpha x}\right)\right)$$



Sine-Gordon model is integrable and can construct higher charge kinks from this basic one.

Derrick's Theorem I

- We just saw two examples of topological solitons and that we could find minimising field configurations by solving first order equations. Why is this not more familiar?
- In dimensions higher than $1 + 1$ it is difficult to find theories with static soliton solutions.
- This is due to a scaling result known as Derrick's Theorem.

Derrick's Theorem II

Theorem (Derrick's Theorem)

Consider a vector, ϕ, constructed from $n-$scalar fields in $1+D$ dimensions,

$$\phi : M^{1+D} \to N^n.$$

Assume that the dynamics of the fields is governed by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - U(\phi),$$

with U non-negative and bounded below. Then for $D > 2$ the only non-singular, finite energy, static solutions are elements of $\mathcal{V}$, the ground states.

Derrick's Theorem proof I

It is worth thinking about the proof of this result. Define

$$V_1 = \frac{1}{2}\int d^D x\, (\nabla\phi)^2\,,$$
$$V_2 = \int d^D x\, U(\phi).$$

$V_i \geq 0$. Next given a static solution $\phi(x)$ we can define the scaled solution

$$\phi_\lambda(\vec{x}) \equiv \phi(\lambda\vec{x}), \qquad \lambda \in \mathbb{R}_+.$$

We then have the scaled potential

$$V_\lambda = \lambda^{2-D} V_1 + \lambda^{-D} V_2.$$

Derrick's Theorem proof II

Critical points are solutions of

$$\left.\frac{dV_\lambda}{d\lambda}\right|_{\lambda=1} = -\left[(D-2)\,V_1 + D\,V_2\right] \stackrel{!}{=} 0.$$

$D > 2$ and $V_i \geq 0$ so the solution is $V_1 = V_2 = 0$.

For $D = 2$ $V_2 = 0$ by the same logic but V_1 is scale invariant and there are non-vacuum solutions.

Another way to state the proof of Derrick's Theorem is that the scaled potential , V_λ is monotonically decreasing as λ increases and thus has no stationary points.

Circumventing Derrick's Theorem

- There are several ways to circumvent Derrick's Theorem and find solitons in higher dimensions.
- Including: Adding higher order terms to the energy, e.g. the Skyrme term or a sextic term.
- Adding terms which are not bounded below, we will encounter this on Wednesday.
- Constructing time dependent solutions.
- Coupling to gauge fields.

Higher dimensional examples

- On Wednesday we will spend our time studying one particular two dimensional example. A recently introduced solvable model of skyrmions in chiral magnets.
- Before doing that we will encounter two other two dimensional models, one gauged the other not.

Abelian-Higgs Model I

- Consider scalar $U(1)$ gauge theory in 2D. Fields are $\phi : \mathbb{R}^2 \to \mathbb{C}$ and connection A.
- The potential is like ϕ^4

$$U(\phi) = \frac{\lambda}{2}\left(|\phi|^2 - a^2\right)^2 .$$

- The vacuum manifold is thus

$$\mathcal{V} = \{\phi = ae^{i\sigma} | \sigma \in \mathbb{R}\} \simeq S^1.$$

- The topology comes from $\pi_1\left(S^1\right) = \mathbb{Z}$, as $\partial\mathbb{R}^2 \simeq S^1_\infty$ and $\phi : S^1_\infty \to \mathcal{V}$, with topological charge

$$n = \frac{1}{2\pi}\oint d\ln\phi = \int_0^{2\pi} \frac{d\sigma}{2\pi}.$$

- Again the topology arises from the boundary condition.

Abelian-Higgs Model II

This model describes vortices in superconductors with magnetic flux $\Phi = \frac{2\pi n}{e}$.

- In general constructing explicit vortex configurations in this model is hard.
- There are several new integrable variations on this model on Riemann surfaces.
- The static energy is

$$E = \frac{1}{2} \int d^2x \left[\frac{B^2}{\Omega} + \overline{D_i\phi} D^i\phi + \frac{\lambda\Omega}{4} \left(1 - |\phi|^2\right)^2 \right],$$

Ω conformal factor of $ds^2 = \Omega \left[(dx)^2 + (dy)^2 \right]$, $D_i\phi = \partial_i\phi - iA_i$ and $B = \partial_1 A_2 - \partial_2 A_1$.

Integrable generalisations of AH I

- In the critically coupled model $\lambda = a^2 = 1$ there are Bogomol'nyi equations
$$B = \frac{\Omega}{2}\left(1 - |\phi|^2\right), \qquad D_1\phi + iD_2\phi = 0.$$
- Such that E bounded by the flux of B, $E \geq \pi N$.
- Can combine the Bogomol'nyi equations into
$$\frac{1}{\Omega}\nabla h + 1 - e^h = 0, \qquad h = 2\log|\phi|.$$
- On Hyperbolic space, with local $\mathbb{C}$ coord z and $\Omega = \frac{8}{(1-|z|^2)^2}$, this is Liouville's equation solved by
$$h = \log\left(\frac{2}{1-|f|^2}\left|\frac{df}{dz}\right|\right)$$
for $f : H^2 \to H^2$ a rational map, See Witten 1977.

Integrable generalisations of AH II

- There are further integrable generalisations corresponding to constant curvature Riemann surfaces

$$B = -\frac{\Omega}{2}\left(K_0 - K|\phi|^2\right), \qquad D_1\phi + iD_2\phi = 0$$

- Here K_0 and K are the Gauss curvature of constant curvature Riemann surfaces.
- For more details see N. Manton Five vortex equations 2016 and A. Popov Integrable vortex-type equations on the two-sphere 2012.

Integrable generalisations of AH III

The integrable cases are

1. $K_0 = K = -1$ Hyperbolic vortices on H^2,
2. $K_0 = K = 1$ Popov vortices on S^2,
3. $K_0 = 0, K = 1$ Jackiw-Pi vortices on $\mathbb{R}^2$,
4. $K_0 = -1, K = 1$ Ambjorn-Olsen vortices on H^2 and,
5. $K_0 = -1, K = 0$ Bradlow vortices on H^2.

The vortex solutions are given by rational maps.
They are also encoded in flat non-Abelian connections valued in three dimensional Lie groups. This direction is explored in CR and B. Schroers 2018.

$O(3)$ sigma model

- The other two dimensional model to briefly mention is the $O(3)$ sigma model.
- The static energy is

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} d^2x\,(\nabla m)^2\,, \qquad m:\mathbb{R}^2 \to S^2$$

- Finite energy solutions extend to $m: S^2 \to S^2$ with topological charge

$$Q[m] = \frac{1}{4\pi}\int d^2x\,(m\cdot\partial_1 m \times \partial_2 m)\,.$$

- This permits a Bogomol'nyi rewriting as

$$E[m] = \frac{1}{2}\int_{\mathbb{R}^2} (\partial_1 m \pm m\times\partial_2 m)^2 + 2\pi|Q[m]|.$$

O(3) Bogomol'nyi equations

- The Bogomol'nyi equations are

$$\partial_1 m \pm m \times \partial_2 m = 0$$

  whose solutions are the minimisers in a given topological sector.
- On Wednesday we will see examples of soliton solutions to these Bogomol'nyi equations, usually called lumps.
- The model of chiral magnets that we will study is a generalisation of the $O(3)$ model which includes extra terms.