Notes related to the String theory and D-Branes reading group

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Hopefully I will tex up notes on some of the topics that we cover during our reading group following [1]. I don't mean to suggest that I will make exhaustive notes or even any notes on most of the discussions. However, I will try and include the important points that we discuss. Other major references were [2, 3] and my notes from [4]. A more technical discussion of the BRST quantisation of a gauge theory is given in [5, 6]. A nice discussion of the operator state correspondence is given in [7]

1 Week 1: 26/1/2017

Mainly discussing chapter 2 of [1] which dealt with the classical string. Questions we discussed included whether the string was smoothly embedded, we think yes it must be, and that the ends of an open string move at the speed of light. We also discussed the different boundary conditions that an open string can have leading to attaching the ends to sub-manifolds of your space time, our first encounter with Branes.

2 Week 2: 2/2/2017

After reading chapter 3 of [1] we had lots of discussions relating to how you impose the constraints. We went through the example of how in light-cone coordinates the operator ordering ambiguity in L_0 for the open string is

$$\epsilon_0 = \frac{d-2}{2}\sum_{n=1}^{n=\infty} n.$$

after zeta function regularisation, see [2] for more details, or other regularisation methods [3] we arrive at

$$\epsilon_0 = -\frac{d-2}{24}.$$

It is not entirely clear to me that all the methods should agree with the sum regularising to $-\frac{1}{12}$. However, I don't know if it is worth spending too much time debating this as I know that the zeta function does analytically continue to this. Philipp and I talked about this possibly being down to picking an on-shell scheme as we know that the "Physical" value of the sum is $-\frac{1}{12}$ from measurements of Casimir energy.
Considering the first exited state of the open string and assuming Lorentz invariance leads us to conclude that $\epsilon_0 = a = 1$ which says that $d = 26$. There are more involved ways of getting this out involving checking that the Lorentz algebra is satisfied but this at least gives a feel for where it arises.
The next topic of discussion was what happens to the central charge when we use the old covariant

method of quantisation. It turns out that this is $\frac{d}{12}$. This time to get to the "Critical dimension" requires considering expectation values of $[L_m, L_{-m}]$ which doesn't lead to an equality but to a bound. This is that for $a \leq 1$ and $d \leq 26$ the string spectrum is free of negative norm states. This is discussed in detail in [2] but at we found it very puzzling why seeking agreement with the lightcone gauge method is sufficient to throw away the $a < 1$, $d < 26$ cases. However, [2] says that these other cases can run in to problems at loop order so rather than there being a unique choice there is a "most natural" choice.
To try and resolve some of our puzzling about getting out the critical dimension and what happens with the conformal anomaly we decided to have a brief interlude to talk about BRST quantisation of the string as the seems to deal with both of these.

3 Week 3: 9/2/2017

3.1 Faddeev Popov ghosts and BRST quantisation.

3/2/2017: These notes were originally written before the discussion and are designed to collect together some of what I know about the Faddeev-Popov procedure and BRST quantisation. Hopefully this will help keep me on track when we come to discuss it. Most of my discussion will be inspired by what I encountered in [4]. However, the notes for this course are not available online so I will be borrowing from my handwritten copy. Updates have been made following the discussion.

3.1.1 The Moral story

Gauge invariance is not a symmetry, that is a misnomer. It is actually a redundancy in our description of a physical system where we are are counting physically equivalent field configurations separately. Classically this is not a problem. However in the path integral this causes us to integrate over the same physical field infinitely many times. To see this consider the space of gauge fields foliated by the orbits of the gauge group, see Figure 1 below for a rough picture of this setup. Everything on the same orbit of the gauge group is a physically indistinguishable configuration so we only want to integrate over one representative of each orbit. To achieve this we gauge fix to a slice of the field space which is transverse to all the orbits. (To do this in general is related to the existence of global section of the gauge bundle. This doesn't always exist so have to work locally and can get Gribbov ambiguities when patching. This wasn't really within the scope of our discussion so is just mentioned here for completeness.) This is achieved by the Faddeev-Popov procedure. It works by introducing auxiliary fields called Faddeev-Popov ghosts and using them to produce a gauge fixing delta function. To spell this out in detail we will start by considering the case of the point particle before moving on to the string.

3.1.2 Point Particle

We are interested in the path integral for a point particle with action,

$$I[x, p; e] = \int dt \big(\dot{x}^i p_i - \frac{1}{2} e(p^2 + m^2)\big),$$

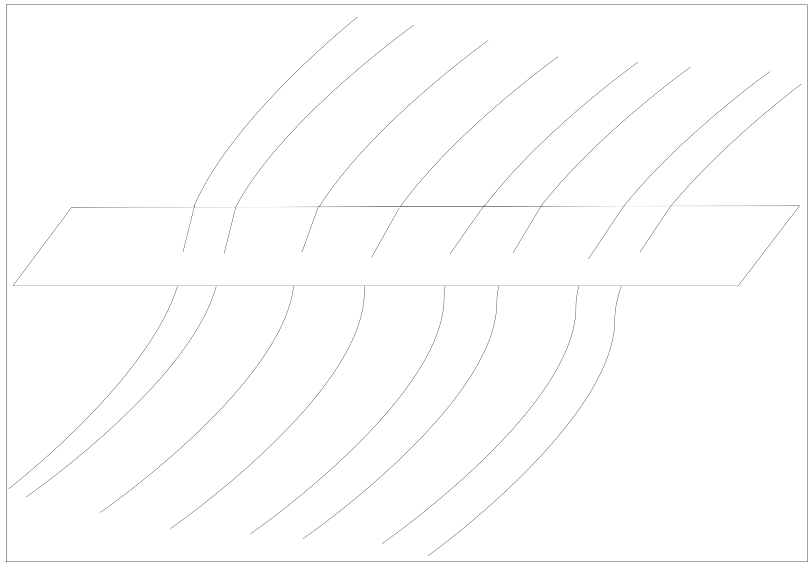

Figure 1: This is a schematic of the space of gauge fields, the curves are orbits of the gauge group and the slice in the middle is what we will gauge fix to. This slice is transverse to all the orbits. The picture here is a local one and there is the potential for the same orbit to intersect the slice at two different points, related to a Gribov ambiguity, when we start trying to patch neighbourhoods together to get a global picture.

where $\vec{x}, \vec{p}, m$, and e are respectively the particles position, conjugate momentum, mass and the Lagrange multiplier imposing the mass shell constraint, $\varphi = \frac{1}{2}(p^2 + m^2) = 0$. e is also the einbein for the particle's worldline and the action is invariant under reparametrisations which are our gauge transformations in this case. The path integral will be summarised as

$$A[x] \propto \int [de] \int [dxdp] e^{iI[x,p;e]},$$

where the constant of proportionality is a, probably infinity, constant factor and $A[x]$ is a quantum mechanical amplitude. It is convenient to work with the Euclidean action so we need to identify $t = -i\tau$ then Wick rotate the action by i to make τ real and relabel it as t ,

$$\begin{aligned} I[x,p;e] &= \int d\tau \{ -i \big(i\dot{x}^i p_i - \frac{1}{2} e(p^2 + m^2) \big) \}, \\ &= \int d\tau \big(\dot{x}^i p_i + \frac{i}{2} e(p^2 + m^2) \big), \\ \Rightarrow iI[x,p;e] &= \int dt \big(i\dot{x}^i p_i - \frac{1}{2} e(p^2 + m^2) \big) = I_E[x,p;e]. \end{aligned}$$

This allows us to write the amplitude as

$$A[x] \propto \int [de] \int [dxdp] e^{-I_E[x,p;e]}.$$

Now we want to deal with the gauge redundancy in e. Ideally we would impose that $e(t) = s$, the proper time. However, we would run in to problems, similar to how imposing this classically requires us to impose the constraints as initial conditions on the particle. Here the problem is that we are implicitly integrating over maps, $\alpha(t)$, which map the gauge group to the world line. The Faddeev-Popov procedure is how we deal with this. Now $e(t)$ is related to s via a gauge transformation,

$$e(t) = s + \dot{\alpha}(t) = e_s[\alpha(t)],$$

This transformation has Jacobian

$$\Delta_{FP} = \det\Big(\frac{\delta e_s[\alpha(t)]}{\delta\alpha(t')}\Big) = \det(\delta(t-t')\partial_t),$$

which we call the Faddeev-Popov determinant. The integral over the gauge parameter e can then be expressed as

$$\int [de] = \int_0^\infty ds \int [d\alpha]\Delta_{FP}.$$

Using the functional analogue of the defining property of the delta function,

$$1 = \int [de]\delta(e(t) - s),$$

we arrive at a useful way to express the number 1,

$$1 = \int_0^\infty ds \int [d\alpha]\Delta_{FP}\delta(e(t) - s).$$

using this in our expression for that amplitude we have that

$$\begin{aligned} A[x] &\propto \int [de]\Big[\int_0^\infty ds \int [d\alpha]\ \Delta_{FP}\ \delta(e(t)-s)\Big]\int [dxdp]e^{-I_E[x,p;e]}, \\ &= \int [d\alpha]\int_0^\infty ds\Delta_{FP}\int [dxdp]e^{-I_E[x,p;e]}. \end{aligned}$$

Now we can see that by dividing by the volume of the gauge group, $\int [d\alpha]$ we can remove the redundancy and arrive at,

$$A[x] \propto \int_0^\infty ds\Delta_{FP}\int [dxdp]e^{-I_E[x,p;e]},$$

which is only over physically distinct fields. A nice way to rewrite this is to recognise that a determinant can be written as a Gaussian integral over anticommuting fields. To this end we introduce fields, b_i, c^i which satisfy

$$\{b_i, b_j\} = 0, \quad \{b_i, c^j\} = 0, \quad \{c^i, c^j\} = 0.$$

Using these new fields we see that

$$\Delta_{FP} = \int [dbdc]\exp\Big[-i\int dt\int dt'(b_i(t)\delta(t-t')\dot{c}^i(t)\Big] = \int [dbdc]\exp\Big[-i\int dt(b_i(t)\dot{c}^i(t)\Big],$$

the i here is because under complex conjugation $ib_i\dot{c}^i$ is real. Using this in the amplitude, and going back to the real time, we have that

$$
\begin{aligned}
A[x] &\propto \int_0^\infty ds \int [dxdp] \int [dbdc] \exp\Big[\{\int dt(i\dot{x}^i p_i - \frac{i}{2}e(p^2+m^2) - b_i(t)\dot{c}^i(t)\Big], \\
&= \int_0^\infty ds \int [dxdp] \int [dbdc] e^{iI_{qu}},
\end{aligned}
$$

where

$$
I_{qu} = \int dt\big(\dot{x}^i p_i + ib_i(t)\dot{c}^i(t) - \frac{1}{2}e(p^2+m^2)\big).
$$

From this new action we can read off the Poisson brackets as

$$
\{x^i, p_j\}_{PB} = -\delta^i_j, \quad \{b_i, c^j\}_{PB} = i\delta^j_i.
$$

There was a nice discussion of how the ghosts can be viewed as giving an extended, graded, phase space in [4]. This starts to make contact with the topic of NQ-Manifolds. However, I feel that this is beyond the scope of what we want to do so am not including it at this point.
There is a hidden symmetry of this action under the following transformation with constant anti-commuting parameter Λ,

$$
\delta_\Lambda x^m = i\Lambda_i c^i p^m, \quad \delta_\Lambda p = 0, \quad \delta_\Lambda b_i = -\frac{1}{2}(p^2+m^2), \quad \delta_\Lambda c = 0.
$$

This is a gauge transformation for x, p. Applying the transformation to the action and pretending that Λ has t-dependence we see that,

$$
\begin{aligned}
\delta I_{qu} &= \int dt\big(+ \frac{d}{dt}\delta_\Lambda x^i p_i + i\delta_\Lambda b_i(t)\dot{c}^i(t)\big), \\
&= \int dt\big(i\dot{\Lambda}_j c^j p^2 + i\Lambda_j \dot{c}^j p^2 - \frac{i}{2}(p^2+m^2)\Lambda_i \dot{c}^i\big), \\
&= i\int dt\big(\dot{\Lambda}_j c^j p^2 + \frac{1}{2}\Lambda_j \dot{c}^j p^2 - \frac{1}{2}m^2\Lambda_i \dot{c}^i\big), \\
&= i\int dt\big(\dot{\Lambda}_j c^j p^2 - \frac{1}{2}\dot{\Lambda}_j c^j p^2 + \frac{1}{2}m^2\dot{\Lambda}_i c^i\big), \\
&= \frac{i}{2}\int dt \dot{\Lambda}_j (p^2+m^2)c^j,
\end{aligned}
$$

where we have made use of the equations of motion, $\dot{p} = 0$ and $\frac{d}{dt}(p^2+m^2) = 0$ during the integration by parts. The Noether charge of this symmetry, suppressing the index on the anticommuting variable, is thus

$$
Q_{BRST} = \frac{1}{2}c(p^2+m^2).
$$

This will generate the above transformation through Poisson brackets. It also satisfies

$$
\{Q_{BRST}, Q_{BRST}\}_{PB} = 0.
$$

The next thing to do is to quantise our action, I_{qu}. Applying minimal prescription, $\{,\}_{PB} \to -i[,]$, we have the commutation relations

$$[\hat{x}^m, \hat{p}_n] = i\delta^m_n, \quad \{\hat{b}_i, \hat{c}^j\} = \delta^j_i,$$

the $\hat{b}$ and $\hat{c}$ operators still anticommute individually. The Hamiltonian operator is then

$$\hat{H} = -\frac{1}{2}(\hat{p}^2 + m^2),$$

and we can define a ghost number,

$$n_{gh} = \hat{c}^i \hat{b}_i,$$

which commutes with $\hat{H}$. Note that

$$[n_{gh}, \hat{c}^j] = \hat{c}^i\hat{b}_i\hat{c}^j - \hat{c}^j\hat{c}^i\hat{b}_i = -\hat{c}^i\hat{c}^j\hat{b}_i + \hat{c}^i\delta^j_i - \hat{c}^j\hat{c}^i\hat{b}_i = \hat{c}^j,$$
$$[n_{gh}, \hat{b}_j] = \hat{c}^i\hat{b}_i\hat{b}_j - \hat{b}_j\hat{c}^i\hat{b}_i = -\hat{c}^i\hat{b}_j\hat{b}_i + \hat{c}^i\hat{b}_j\hat{b}_i - \delta^i_j\hat{b}_i = -\hat{b}_j,$$

which we interpret as saying that the c-ghosts have ghost number 1 and the b-ghosts have ghost number -1. We also see that Q_{BRST} will become the operator

$$\hat{Q}_{BRST} = \frac{1}{2}\hat{c}(\hat{p}^2 + m^2),$$

which has ghost number 1. Also $\hat{Q}^2_{BRST} = 0$ and note that

$$\hat{H} = [e\hat{b}, \hat{Q}_{BRST}].$$

Now we take as our physical state condition that

$$\hat{Q}_{BRST}|\text{phys}\rangle = 0,$$

which can be motivated by seeing that

$$\langle \text{phys'}|\hat{H}|\text{phys}\rangle = 0.$$

This is because $\hat{H}$ is not gauge invariant and we want to see that the physical state implies that non-gauge invariant operators have only zero expectation values. If we have an inner product such that $\hat{Q}_{BRST}$ is hermitian then we can show that

$$||\hat{Q}_{BRST}|\text{phys}\rangle||^2 = 0,$$

which means that we should define physical states as equivalence classes,

$$|\Psi\rangle \sim |\Psi\rangle + \hat{Q}_{BRST}|\chi\rangle,$$

for any state $|\chi\rangle$. To make more sense of this we look at operators which realise the anticommutation relations,

$$\hat{p}_m = -i\partial_m, \quad \hat{b} = \frac{\partial}{\partial c},$$

and $\hat{x}, \hat{c}$ are diagonal with eigenvalues x, c.
Using this

$$\hat{Q}_{BRST} = -\frac{c}{2}(\Box - m^2),$$

which means that the physical state condition becomes that the particles wave function satisfies the Klein-Gordon equation. To avoid the ghost states being physical states we also impose that

$$b|\text{phys}\rangle = 0.$$

Exercise for the Reader: This procedure can be repeated, and it was in [4], for the case that

$$I = \int dt\big(\dot{q}^i p_i - \lambda^i \phi_i\big),$$

where $\vec{q}$ is a generalised coordinate, λ^i are Lagrange multipliers and the ϕ_i are first class constraints satisfying

$$\{\phi_i, \phi_j\}_{PB} = f_{ij}{}^k \phi_k.$$

Here the gauge condition we want to impose is that $\lambda^i = \bar{\lambda}^i$ and $\delta_\epsilon \lambda^i = \dot{\epsilon}^i + \epsilon^j \bar{\lambda}^k f_{jk}^{\ \ i}$ and the Faddeev-Popov determinant is

$$\Delta_{FP} = \det\big[\delta(t-t')(\delta^i_j \partial_{t'} + \bar{\lambda}^k f_{jk}^{\ \ i})\big].$$

The steps are the same just slightly more care needs to be taken as now there will be a ghost term in the Hamiltonian coming from the second term in the Faddeev-Popov determinant.

3.1.3 Bosonic String

Now that we know what the story is for the point particle we can take the step up to considering the string. In the case of the point particle it was the reparametrisation invariance of the worldline that we "gauged" for the string it will be the residual conformal invariance. This is easiest to see if we work with the Nambu-Goto string in light cone coordinates where we can impose the conformal gauge on the Lagrange multipliers, $\lambda^\pm = 1$. In this gauge the variation of the multipliers is that

$$\delta\lambda^\pm = \mp\sqrt{2}\partial_\mp \xi^\pm.$$

From this we can read off that the Jacobian of the transformation will be

$$\Delta_{FP} = \det\begin{pmatrix} \sqrt{2}\delta(t-t')\delta(\sigma-\sigma')\partial'_+ & 0 \\ 0 & -\sqrt{2}\delta(t-t')\delta(\sigma-\sigma')\partial'_- \end{pmatrix},$$

as this determinant will just be the product of two terms the contribution to the path integral will involve four types of ghost. In fact we can see that

$$\begin{aligned}\Delta_{FP} &= \int [dbdcd\tilde{b}d\tilde{c}] \exp\Big[-\sqrt{2}\int dtdt'd\sigma d\sigma'\big(b\delta(t-t')\delta(\sigma-\sigma')\partial'_+ c' - \tilde{b}\delta(t-t')\delta(\sigma-\sigma')\partial'_- \tilde{c}'\big)\Big], \\ &= \int [dbdcd\tilde{b}d\tilde{c}] \exp\Big[-\sqrt{2}\int dt \oint d\sigma\big(b\partial_+ c - \tilde{b}\partial_- \tilde{c}\big)\Big].\end{aligned}$$

This means that our path integral for the string with action

$$I[X,P;\lambda] = \int dt \oint d\sigma\big(\dot{X}^m P_m - \lambda^{\pm}\varphi_{\pm}\big),$$

will be of the form,

$$\begin{aligned} A[X] &\propto \int [dXdPdbdcd\tilde{b}d\tilde{c}] \exp\Big[i\int dt \oint d\sigma\big(\dot{X}^m P_m - H(X,P)\big) - \sqrt{2}\int dt \oint d\sigma\big(b\partial_+ c - \tilde{b}\partial_- \tilde{c}\big)\Big], \\ &= \int [dXdPdbdcd\tilde{b}d\tilde{c}] \exp\Big[i\int dt \oint d\sigma\big(\dot{X}^m P_m - H(X,P) + i\sqrt{2}b\partial_+ c - i\sqrt{2}\tilde{b}\partial_- \tilde{c}\big)\Big], \\ &= \int [dXdPdbdcd\tilde{b}d\tilde{c}] \exp\Big[i\int dt \oint d\sigma\big(\dot{X}^m P_m + ib\dot{c} + i\tilde{b}\dot{\tilde{c}} - H(X,P) + i(bc' - \tilde{b}\tilde{c}')\big)\Big], \\ &= \int [dXdPdbdcd\tilde{b}d\tilde{c}] \exp\Big[i\int dt \oint d\sigma\big(\dot{X}^m P_m + ib\dot{c} + i\tilde{b}\dot{\tilde{c}} - H_{qu}\big)\Big], \end{aligned}$$

where we have used that $\sigma^{\pm} = \frac{1}{\sqrt{2}}(\sigma \pm t)$, $H(X,P) = \frac{P^2}{2T} + \frac{T}{2}(X')^2$ is the string Hamiltonian in conformal gauge and

$$H_{qu} = H(X,P) - i(bc' - \tilde{b}\tilde{c}'),$$

is the Hamiltonian including the ghost contribution.

It is pointed out in [4] that this action is conformally invariant with the ghost fields transforming as

$$\delta_{\xi} c = \xi^- \partial_- c - \partial_- \xi^- c,$$

a vector field and

$$\delta_{\xi} b = \xi^- \partial_- b + 2\partial_- \xi^- b,$$

a quadratic differential with $(\tilde{b}, \tilde{c})$transforming similarly. The way that the b fields transform is why in some sources, notably [2, 3], they are shown with two space time indices. Following [4] we will suppress these indices but will try to remember that they are there. The Noether charges for this symmetry give the components of the stress energy tensor,

$$\Theta^{gh}_{--} = -\frac{i}{\sqrt{2}}(2b\partial_- c - c\partial_- b), \quad \Theta^{gh}_{++} = \frac{i}{\sqrt{2}}(2\tilde{b}\partial_+ \tilde{c} - \tilde{c}\partial_+ \tilde{b}).$$

It is essential to have these charges at this point as when Fourier transformed they will give us the constraints for the ghost fields. From the action we can read off the Poisson brackets for the ghost fields as being,

$$\{b(\sigma), c(\sigma')\}_{PB} = -i\delta(\sigma - \sigma') = \{\tilde{b}(\sigma), \tilde{c}(\sigma')\}_{PB}.$$

Now we are interested in the Fourier space action for the open string. To does this Fourier expand the ghost fields as

$$\begin{aligned} c &= \sum_{k\in\mathbb{Z}} e^{ik\sigma} c_k, \quad b = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ik\sigma} b_k, \\ \tilde{c} &= \sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{c}_k, \quad \tilde{b} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{b}_k \end{aligned}$$

and substitute into the action. At this point I'm going to cheat and not do the Fourier transformation of the ordinary field part as we all ready know the answer. After all this has been done the action takes the form

$$I_{qu} = \int dt \Big(\dot{x}^m p_m + \sum_{k=0}^{\infty} \frac{i}{k} (\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \tilde{\alpha}_{-k}) + \sum_{n\in\mathbb{Z}} i(b_{-n}\dot{c}_n + \tilde{b}\dot{\tilde{c}}_n) - H_{qu} \Big),$$

where the "quantum" Hamiltonian is of the form

$$H_{qu} = L_0 + N_{gh} + \tilde{L}_0 + \tilde{N}_{gh}.$$

Here we have written the ghost level number as

$$N_{gh} = \sum_{n=1}^{\infty} k(b_{-k}c_k + c_{-k}b_k),$$

and similarly for the $\sim$ fields. From this action we can read off the ghost Poisson brackets as

$$\{c_n, b_{-n}\}_{PB} = -i = \{\tilde{c}_n, \tilde{b}_{-n}\}_{PB}.$$

The next step is to finally fulfil our promise of Fourier transforming the ghost Noether charges to get

$$L_m^{gh} = \sum_{n\in\mathbb{Z}} (m+n) b_{n-k} c_k, \quad \tilde{L}_m^{gh} = \sum_{n\in\mathbb{Z}} (m+n) \tilde{b}_{n-k} \tilde{c}_k,$$

from which we observe that

$$L_0^{gh} = N_{gh}, \quad \tilde{L}_0^{gh} = \tilde{N}^{gh},$$

and can calculate that

$$\{L_m^{gh}, c_n\}_{PM} = i(2m+n)c_{n+m}, \quad \{L_m^{gh}, b_n\}_{PM} = -i(m-n)b_{n+m},$$

which leads to the algebra obeyed by the L_m^{gh} being the familiar

$$\{L_m^{gh}, L_n^{gh}\}_{PB} = -i(m-n)L_{m+n}^{gh}$$

again the $\sim$ variables are similar.

Exercise for the Reader: Check the details of the calculation up to this point. Laziness on my part has stopped me from including it and it is a worth while exercise to do once!

This is the Witt algebra that the usual constraints, L_m, also satisfy so including both types of constraint we have Witt $\bigoplus$ Witt. This means that we can combine the constraints as

$$\mathcal{L}_m = L_m + L_m^{gh}, \quad \tilde{\mathcal{L}}_m = \tilde{L}_m + \tilde{L}_m^{gh},$$

with Poisson brackets,

$$\begin{aligned}
\{\mathcal{L}_m, \mathcal{L}_n\}_{PB} &= -i(m-n)\mathcal{L}_{m+n}, \\
\{\mathcal{L}_m, \tilde{\mathcal{L}}_n\}_{PB} &= 0, \\
\{\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n\}_{PB} &= -i(m-n)\tilde{\mathcal{L}}_{m+n}.
\end{aligned}$$

At this point [4] makes a very nice point about reading off the conformal weight, J, of an operator from its Poisson bracket with $\mathcal{L}_m$,

$$\{\mathcal{L}_m, \mathcal{O}^J_n\}_{PB} = -i[m(J-1)-n]\mathcal{O}^J_{m+n}.$$

From this we can see that α_k has $J=1$, c_n has $J=-1$ and b has $J=2$ and similarly for the $\sim$ fields. Now that we have the algebra we can quatise. We know that there is a central charge of $\frac{d}{12}$ appearing in the commutation relations for our L_m,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}(m^3-m)\delta_{m,-n}.$$

we know that the L^{gh}_m will obey something similar but not yet what the central charge will be,

$$[L^{gh}_m, L^{gh}_n] = (m-n)(L^{gh}_{m+n} - a\delta_{m,-n}) + \frac{c_{gh}}{12}(m^3-m)\delta_{m,-n},$$

where the a term appears by considering the zero modes. In [2] they deal with this by defining $\mathcal{L}_m = L_m + L^{gh}_m - a\delta_m$ so that classically the zero'th constraint is that $\mathcal{L}_0 = 0$, this is just a way of accounting for the ordering ambiguity in $\mathcal{L}_0$. We can see this by the same calculation that we saw used in the case of the L_m. Now to find c_{gh} we consider the case $n=-m$ and taking expectation values of the ghost vacuum. We use the physical state condition $L^{gh}_m|0\rangle = 0$ with $m \geq 0$ to arrive at

$$||L^{gh}_{-m}|0\rangle||^2 = -2ma + \frac{c_{gh}}{12}(m^3-m).$$

For calculational ease we will restrict to the cases $m=1,2$. Here we have

$$||L^{gh}_{-1}|0\rangle||^2 = -2a, \quad ||L^{gh}_{-2}|0\rangle||^2 = -4a + \frac{c_{gh}}{2}.$$

We can also calculate these directly from our above expression for L^{gh}_m and the, perfectly reasonable, assumption that $b_k|0\rangle = c_k|0\rangle = 0$ for $k>0$;

$$\begin{aligned}
L^{gh}_{-1}|0\rangle &= \sum_{k\in\mathbb{Z}}(k-1)b_{-1-k}c_k|0\rangle = -(2b_0c_{-1} + b_{-1}c_0)|0\rangle, \\
L^{gh}_{-2}|0\rangle &= \sum_{k\in\mathbb{Z}}(k-2)b_{-1-k}c_k|0\rangle = -(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|0\rangle.
\end{aligned}$$

To calculate the expectations we also need to assume, just like is done for the α_k that the positive and negative k oscillators are related via $b^\dagger_k = b_{-k},\ c^\dagger_k = c_{-k}$. This is just the statement that our operators are hermitian with respect to the given inner product of states. Once we have this we can calculate that

$$\begin{aligned}
||L^{gh}_{-1}|0\rangle||^2 &= \langle 0|(2c_1b_0 + c_0b_1)(2b_0c_{-1} + b_{-1}c_0)|0\rangle, \\
&= 2\langle 0|(c_1b_0b_{-1}c_0 + c_0b_1b_0c_{-1})|0\rangle, \text{ using } b_0^2 = c_0^2 = 0, \\
&= 2\langle 0|(-b_0c_0c_1b_{-1} - c_0b_0b_1c_{-1})|0\rangle, \text{ using that the operators anticommute }, \\
&= 2\langle 0|(-b_0c_0\{c_1, b_{-1}\} - c_0b_0\{b_1, c_{-1}\})|0\rangle, \text{ using that } c_1, b_1 \text{ annihilate the vacuum,}, \\
&= -2\langle 0|(b_0c_0 + c_0b_0)|0\rangle \text{ since } \{b_1, c_{-1}\} = 1, \\
&= -2,
\end{aligned}$$

However we saw above that this was $-2a$ which implies that $a = 1$. Next we do the same with L^{gh}_{-2} to find c_{gh}, this time the identities used will not be pointed out at every step to conserve space but they will be the same as those used above.

$$\begin{aligned}
||L^{gh}_{-2}|0\rangle||^2 &= \langle 0|(2c_0b_2 + 3c_1b_1 + 4c_2b_0)(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|0\rangle, \\
&= \langle 0|(-8c_0b_0b_2c_{-2} + 9c_1b_1b_{-1}c_{-1} - 8c_2b_{-2}b_0c_0)|0\rangle, \\
&= -8\langle 0|(c_0b_0\{b_2, c_{-2}\} + \{c_2, b_{-2}\}b_0c_0|0\rangle - 9\langle 0|c_1b_{-1}\{b_1, c_{-1}\}|0\rangle, \\
&= -8\langle 0|\{c_0, b_0\}|0\rangle - 9\langle 0|\{c_1, b_{-1}\}|0\rangle, \\
&= -8 - 9, \\
&= -17,
\end{aligned}$$

which we combine with the above result to see that,

$$-17 = -4 + \frac{c_{gh}}{2}, \quad \Rightarrow \; c_{gh} = -26.$$

Going to the full algebra for the $\mathcal{L}_m$'s we see that

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)(L^{gh}_{m+n} - \delta_{m,-n}) + \frac{d - 26}{12}(m^3 - m)\delta_{m,-n},$$

so the conformal anomaly vanishes in $d = 26$. This is in agreement with what we saw in light cone gauge quantisation where preserving Lorentz invariance tells us that $a = 1$ and $d = 26$. Here non-vanishing of the conformal anomaly would seem to lead to negative norm-states that can satisfy the physical state conditions where as in the old-covariant case even with a non-zero central charge all the negative norm states were un-physical for $D \leq 26$. There is probably more to say about this but I don't know what that is.

Finally we are ready to consider the BRST quantisation of the Nambu-Goto string. While originally it was hoped to do this without going to light-cone coordinates these were the most natural for considering the gauging of the conformal symmetry above and so we will continue to use them. It turns out that if almost every resource uses a particular coordinate system then there is usually a very good reason for this. We saw above that the BRST charge was the c field multiplied by the constraints. This time we will have two charges corresponding to the left and right moving fields,

$$Q_{BRST} = Q_+ + Q_-,$$

for

$$Q_- = \oint d\sigma(cH_- + icc'b), \quad Q_+ = \oint d\sigma(\tilde{c}H_+ - i\tilde{c}\tilde{c}'\tilde{b}).$$

These just follow from considering the constraints in the action with

$$H_\pm = \frac{1}{4T}(P \pm TX')^2,$$

for T the string tension related to α' in the notation of [1]. They satisfy the Poisson brackets,

$$\{H_\pm(\sigma), H_\pm(\sigma')\}_{PB} = \pm(H_\pm(\sigma) + H_\pm(\sigma'))\delta(\sigma - \sigma'), \quad \{H_+(\sigma), H_-(\sigma')\}_{PB} = 0.$$

Using this we arrive at

$$\begin{aligned}
\{Q_+, Q_-\}_{PB} &= 0, \\
\{Q_\pm, Q_\pm\}_{PB} &= 0, \\
\Rightarrow \{Q_{BRST}, Q_{BRST}\}_{PB} &= 0.
\end{aligned}$$

I have checked this but there could be more details presented here. We want to Fourier transform the $Q_\pm$. The first term is easy and gives

$$\sum_{n\in\mathbb{Z}} c_{-n} L_n.$$

The second is more challenging and involves some jiggery-pokery with summation labels.

$$\begin{aligned}
i\oint \frac{d\sigma}{2\pi} \sum_{k,m,l\in\mathbb{Z}} il e^{i(k+l+m)\sigma} c_k c_l b_m &= -\sum_{k,l,m\in\mathbb{Z}} \delta_{k+l+m,0} c_k c_l b_m, \\
&= \sum_{k,m\in\mathbb{Z}} (k+m) c_k c_{-k-m} b_m, \\
&= \frac{1}{2}\Big(\sum_{k,m\in\mathbb{Z}} (k+m) c_k c_{-k-m} b_m + \sum_{k,m\in\mathbb{Z}} (k+m) c_k c_{-k-m} b_m \Big), \\
&= \frac{1}{2}\Big(\sum_{k,m\in\mathbb{Z}} (-k+m) c_{-k} c_{-k-m} b_m - \sum_{k,m\in\mathbb{Z}} (k+m) c_{-k-m} c_k b_m \Big), \\
&= \frac{1}{2}\Big(\sum_{k,m\in\mathbb{Z}} m c_{-k} c_{-m} b_{m+k} - \sum_{k,m\in\mathbb{Z}} (-k+m) c_{k-m} c_{-k} b_m \Big), \\
&= \frac{1}{2}\Big(\sum_{k,m\in\mathbb{Z}} m c_{-k} c_{-m} b_{m+k} - \sum_{k.m\in\mathbb{Z}} m c_{-m} c_{-k} b_{m+k} \Big), \\
&= \frac{1}{2}\Big(\sum_{k,m\in\mathbb{Z}} m c_{-k} c_{-m} b_{m+k} - \sum_{k.m\in\mathbb{Z}} k c_{-k} c_{-m} b_{m+k} \Big), \\
&= -\frac{1}{2} \sum_{k,m\in\mathbb{Z}} (k-m) c_{-k} c_{-m} b_{m+k}.
\end{aligned}$$

This tells us that

$$Q_- = \sum_{n\in\mathbb{Z}} \Big(L_m + \frac{1}{2} L_m^{gh} \Big) c_{-n},$$

and similarly for Q_+ in terms of the $\sim$ fields. Using this we can show that the Hamiltonian is generated by Q_{BRST}. The exact details of this are worth checking! In fact we have that $Q_\pm$ generate the constraints as

$$\mathcal{L}_m = i\{b_m, Q_-\}_{PB}, \qquad \text{and similarly for } \tilde{\mathcal{L}}_m.$$

We are now at the point to quantise. W can read off the canonical anticommutators of the ghost operators as,

$$\{c_n, b_{-n}\} = 1, \quad \{\tilde{c}_n, \tilde{b}_{-n}\} = 1.$$

We also need to define the ghost vacuum as as a tensor product of a left and right vacuum,

$$|0\rangle = |0\rangle_L \otimes |0\rangle_R,$$

where $c_n|0\rangle_R = b_m|0\rangle_R = 0$ for $n > 0, m \geq 0$ and the left vacuum is annihilated by the $\sim$ fields for positive n. This means that the negative n ghost operators are creation operators and the positive n ones are annihilation operators. This means that the full oscillator vacuum will be the product of the ghost vacuum and the ordinary field vacuum and confusingly we will also label it as $|0\rangle$. There is an ordering ambiguity in L_0 so we will choose

$$L_0^{gh} = \sum_{k=1}^{\infty} k(b_{-k}c_k + c_{-k}b_k) = N_{gh},$$

such that $N_{gh}|0\rangle = 0$. This is where the difference between the ghost algebra and the ordinary field algebra comes in. There is not an ordering ambiguity in any of the other constraints so we have that

$$L_m^{gh}|0\rangle = 0,$$

for $m \geq 1$. The N_{gh} and its tilded counterpart are have as eigenvalues the ghost number. There will be an ordering ambiguity in Q_-, due to the one in L_0. This means that in the $n = 0$ component we need to subtract off a factor of a. Taking account of this we have that

$$Q_-|0\rangle = 0 \Rightarrow (L_0 - a)|0\rangle = 0,$$

and that

$$\{b_m, Q_-\} = \mathcal{L}_m - a\delta_{m,0},$$

and

$$[\mathcal{L}_m, b_n] = (m - n)b_{n+m}.$$

Is it worth including the expression for Q_- here for clarity? Need to check that $Q^2 = 0$ is equivalent to the vanishing of the conformal anomaly. Consider;

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= [\{b_m, Q_-\}, \mathcal{L}_n], \\
&= -\{[\mathcal{L}_n, b_m], Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\}, \\
&= (m - n)\{b_{m+n}, Q_-\} + \{[Q_-, \mathcal{L}_n], b_m\}.
\end{aligned}$$

Now as Q_- commutes with $\delta_{m,0}$ we see that

$$[Q_-, \mathcal{L}_m] = [Q_-, \{b_n, Q_-\}] = [Q_-^2, b_n],$$

using the super-Jacobi identity. This tells us that,

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)(\mathcal{L}_{m+n} - a\delta_{m+n}) + \{[Q_-^2, b_n], b_m\}.$$

We see from this that if $Q_-^2 = 0$ the conformal anomaly vanishes. We now want to show the implication the other way around. The common approach to this, [2, 4], is to say that if $Q_-^2 \neq 0$ then it must be quadratic in oscillators as the quartic part vanishes at the classical level and there are no ordering ambiguities To see this need to stare at the form of Q_- for a while. This means that

$$Q_-^2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} c_n c_{-n} A(n),$$

for some function $A(n)$. This then leads to

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)(\mathcal{L}_{m+n} - a\delta_{m+n}) + A(m)\delta_{m,-n},$$

So $A(m)$ is the conformal anomaly and its vanishing implies that $Q_-^2 = 0$. So we have the result that we want. This tells us that $Q_{BRST}^2 = 0$ is equivalent to the vanishing of the conformal anomaly in both parts of the algebra. This means that we can impose

$$Q_{BRST}|\Psi\rangle = 0,$$

as the physical state conditions. Checking using the form of Q_- we can see that $Q_-|\text{phys}\rangle_R = 0$ implies the usual Virasoro constraints that we know and love,

$$\begin{aligned}(L_0 - a)|\Psi\rangle_R &= 0,\\ L_m|\Psi\rangle_R &= 0,\end{aligned}$$

for $m > 0$. Is it worth filling in more of the details here?
This is everything that we wanted to achieve.

3.1.4 Other Occurrences

While we have seen a nice example of BRST quantisation in the case of both the point particle and the string it is worth mentioning some other places where it is relevant. It is often the procedure of choice when quantising a gauge theory as it can allow you to make sense, at least as much as ever can be, of the integral over the gauge field. The most common place to see it would be when quantising a Yang-Mills theory. The procedure their is the conceptually the same as we saw above however some of the details will differ as the gauge group is typically $SU(2)$ in those cases, though it can be any unitary group. To see what happens in this case just consult your favorite QFT text book.

3.2 What We Actually Discussed

We ended up jumping the gun and talking about how interactions could be introduced as well as why we we are only considering a first quantised theory as opposed to a string field theory. We we did talk about Faddeev-Popov and BRST we realised that there was a point of confusion over how the gauge fixing term is introduced in the Path integral. This was around the fact that we should really introduce the gauge fixing delta function for a dummy variable $\tilde{e}(t)$ but we are performing the functional integral over $e(t)$. Subsequently a couple of us realised that what we are doing is substituting in the $\tilde{e}(t)$ gauge fixing, gauge transforming $e \to \tilde{e}(t)$, using that the action is invariant under this transformation and then relabelling $\tilde{e}(t)$ as $e(t)$.

4 Week 4: 17/2/2017

We discussed the motivation for introducing sting interactions. The case of the graviton was the one we started to understand, where we think of embedding the string in a spacetime with a metric perturbed away from Minkowski. If we expand in terms of the perturbation we see operator insertions in the path integral corresponding to the vertex operator for a graviton. This lead on to a discussion of how to contruct the operator associated to a given state. The main point, following [7], was that, considering Euclidean signature closed strings for simplicity, the oscillator modes can be expressed in terms of the sum of right and left moving fields as;

$$\alpha^{\mu}_{-n} = \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \oint \frac{dz}{2\pi} z^{-n} \partial_z X^{\mu}(z),$$
$$\tilde{\alpha}^{\mu}_{-n} = \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^{-n} \partial_{\bar{z}} X^{\mu}(\bar{z}).$$

When states involving more than one oscillator mode are considered we will encounter expressions like

$$\alpha^{\nu}_{-n}\alpha^{\mu}_{-m} = \frac{2}{\alpha'} \oint \frac{dzdw}{(2\pi)^2} z^{-n} w^{-m} \partial_z X^{\nu}(z) \partial_w X^{\mu},$$

which can be made sense of using the operator product expansion. More details are given in Section 4 of [3]. Hopefully we gained some intuition for this even though we stopped short of explicitly using the OPE.

We also discussed how vertex operators for the open string will be related to integrals over the boundary of the worldsheet. This follows in a similar way to the above as when we map to the upper half-plane, (Looks like we have removed the point $z = 0$ which corresponds to $\tau = -\infty$ as the map is not injective there. In fact it sends the whole string to that one point. This is the point that the operators will inserted at.), the oscillators modes are represented as integrals over the boundary.

5 Week 5: 3/3/2017

Hopefully we will spend some time talking about the Virasoro-Shapiro amplitude for closed string tachyon scattering so that we can get an idea of an explicit calculation. Then we can move on to discussing the Superstring.

I wasn't there for all of the discussion so someone else will be contributing to the notes. While I was there we were interested in the fact that the sum over metrics only makes sense for the Euclidean path integral as there are restrictions on the existence of Lorentzian metrics on Riemann surfaces. This raises the question of the legitimacy of Wick rotating so that we can make sense of the path integral. However, this would seem to be ok as for both the closed string world sheet, a cylinder, and the open string worldsheet, a strip, there is a global Lorentzian "metric" so Wick rotation should be ok as long as the world sheet is flat.

6 Week 6:10/3/2017

We encountered the world sheet action for the superstring for the first time and explicitly checked its invariance under a supersymmetry transformation. We then discussed the Fourier modes in the Ramond and Neveu-Schwarz cases as well as the extension of the Virasoro algebra to a super-Virasoro algebra. An interesting observation to make at this point is that the L operator, the Fourier modes of the stress-energy tensor, are of the same form as $L_n + L_n^{gh}$ that we saw above. The key difference is that the fermion Fourier modes have a different conformal weight than the ghost Fourier modes above. The explicit computation of the invariance of the action

$$S = -\frac{T}{2}\int_\Sigma d\tau d\sigma \left(\partial_a x^\mu \partial^a x_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right)$$

under the transformation

$$\delta_\varepsilon x^\mu = \bar{\varepsilon}\psi^\mu, \qquad \delta_\varepsilon \psi^\mu = -i\rho^a \varepsilon \partial_a x^\mu$$

depends on the representation of the gamma matrices, ρ^a, being used. What we encountered seemed to suggest that when the the ρ^a are purely imaginary the ψ can be taken to be Majorana-Weyl and ε is anti-Majorana-Weyl. When the ρ are in a real representation of $Cl(2)$ this will be the other way around.

7 Week 7: 17/3/2017

We used the quick method to arrive at the critical dimension , $d = 10$, for the string in lightcone gauge and discussed the more involved method, checking closure of the Lorentz group as is done in [2]. An interesting coincidence is that in the Ramond sector, the fermionic sum is over integers, we have no ordering ambiguity in L_0. This is because the ambiguity coming from the bosonic and fermionic sums enter with opposite signs but the same terms and lead to an order by order cancellation. This is related to the impact parameter, a_R being zero to preserve Lorentz invariance in the spectrum. Interestingly this tells us that our usual approach to getting out the critical dimension doesn't work in the Ramond sector. In the Neveu-Schwarz sector we get different contributions from the fermionic and the bosonic oscillators and it is instructive to see the calculation. The short cut method starts by considering L_0 in Lightcone gauge,

$$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha^\mu_{-k}\alpha^\nu_k g_{\mu\nu} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\psi_{-r}\cdot\psi_r.$$

The lightcone gauge condition is that $\alpha^+ = 0 = \psi^+$ and once we impose the vanishing of the L_n for $n \geq 1$ we can also eliminate α^- and ψ^- so in the above we are only including the $D-2$ indices $\mu, \nu = 2, \cdots D-1$. An important point to be aware of is that the fermionic sum is over "odd" fractions, $\frac{t}{2}$ where t is an odd integer, and does not include any integers. This was something that we forgot the importance off and lead us to be confused when it came to relabelling the sum over fermionic modes. We can split the bosonic sum and use the commutation relations

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n+m,0}g^{\mu\nu},$$

to get

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \alpha^\mu_{-k}\alpha^\nu_k g_{\mu\nu} &= \alpha_0^2 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k + \sum_{k=-\infty}^{-1}\alpha_k\cdot\alpha_{-k} - (D-2)\sum_{k=-\infty}^{-1} k,\\
&= \alpha_0^2 + 2\sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k + (D-2)\sum_{k=0}^{\infty}(k+1),\\
&= \alpha_0^2 + 2\sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k - \frac{(D-2)}{12}.
\end{aligned}$$

The choice to set take $k \to k+1$ to include the zero term in the sum was made so that we had the same expression as that in [4], this means that we are regularising the generalised zeta function,

$$\zeta(s,q) = \sum_{n=0}^{\infty}(n+q)^{-s},$$

which can be analytically continued to have the value

$$\zeta(-1,q) = -\frac{1}{12}(6q^2 - 6q + 1).$$

As the term we are adding in this case is zero we get the same result in the bosonic case where the regular zeta function is used. Another reason for wanting to use the generalised zeta function is that we have seemingly no choice but to use for the fermionic ordering ambiguity. For the fermions we proceed in the same way but use the anticommutation relations

$$\{\psi^\mu_s, \psi^\nu_r\} = \delta_{s+r,0}g^{\mu\nu}.$$

The calculation proceeds as

$$\begin{aligned}
\sum_{r\in\mathbb{Z}+\frac{1}{2}} r\psi_{-r}\cdot\psi_r &= \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r + \sum_{r=-\infty}^{-\frac{1}{2}} r\psi_{-r}\cdot\psi_r,\\
&= \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - \sum_{r=-\infty}^{-\frac{1}{2}} r\psi_r\cdot\psi_{-r} + (D-2)\sum_{r=-\infty}^{-\frac{1}{2}} r,\\
&= 2\sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - (D-2)\sum_{r=\frac{1}{2}}^{\infty} r,\\
&= 2\sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - (D-2)\sum_{k=0}^{\infty}(k+\frac{1}{2}),\\
&= 2\sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r}\cdot\psi_r - \frac{(D-2)}{24}.
\end{aligned}$$

Hopefully all the relabeling in the final sum is clear and we can see that we definitely need to use the generalised zeta function as $k + \frac{1}{2}$ can not be reinterpreted in terms of the usual zeta function. The particular value that we used here is

$$\zeta(-1, \frac{1}{2}) = \frac{1}{24}.$$

Going back to L_0 we have that

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k + \sum_{r=\frac{1}{2}}^{\infty} r\psi_{-r} \cdot \psi_r + \epsilon_0,$$

where

$$\epsilon_0 = -\frac{D-2}{24}(1 + \frac{1}{2}) = -\frac{D-2}{16}.$$

From considering the level $N_f = \frac{1}{2}$ excitations,

$$\psi^{\mu}_{\frac{1}{2}}|0\rangle,$$

which have to massless to preserve Lorentz invariance we get that

$$\epsilon_0 = -\frac{1}{2}$$

which results in the critical dimension being

$$D = 10.$$

Next we looked at the spectrum for open and closed strings and discussed the motivation of the GSO projection. The most satisfying explanation seemed to be to jump ahead and see that GSO is equivalent to modular invariance at the one loop level which we will explore in more detail next week.

8 Week 8: 24/3/2017

We discussed section 4.5 in [1] where the one-loop vacuum amplitude for the superstring is calculated. The trace computations in Exercise 4.3 were discussed and we went through the details, the bosonic case is also given in [7]. We then spent a long time discussing the different boundary conditions on the fermions and how that affected the partition function, including seeing which combinations were modular invariant and how this is equivalent to imposing GSO. Combining all the details we saw that the fermionic contribution vanishes and thus the total amplitude is zero. Someone else is texing up some of the details of this calculation and it will be included here when it is done.

9 Week 9: 31/3/2017

This week we focussed our discussion on the Ramond-Ramond sector of the open string. Here the different choice of GSO projection for the left and right movers leads to two different string theories; Type IIA where the ground states have the opposite GSO projection imposed on left and right movers and Type IIB where the same GSO projection is used. This mean sthat the ground states form different representations of Spin(9,1). We can decompose the spinor representations into antisymmetric tensor representations. These antisymmetric tensors are n-form analogues of the Maxwell field strength 2-form and due to the properties of the gamma matrices in $Cl(9,1)$ we have dual $10-n$ forms as well. These n-forms and their dual objects satisfy higher dimensional analogues of Maxwell's equations,

$$\partial_{[\nu} F^{(n)}_{\mu_1 \cdots \mu_n]} = 0, \quad \partial^{\nu} F^{(n)}_{\nu \mu_2 \cdots \mu_n} = 0,$$

where for comparison Maxwell's equations are

$$dF^{(2)} = 0 = d \star F^{(2)}.$$

These n-form fields strengths have potentials, C such that

$$F^{(n)}_{\mu_1 \cdots \mu_n} = \partial_{[\mu_1} C^{(n-1)}_{\mu_2 \cdots \mu_n]}.$$

The potentials do not couple to the string[1] and to have a standard coupling we would need to integrate $C^{(n-1)}$ along an $n-1$ dimensional hyper-surface. These hypersurfaces would be non-perturbative states that are charged under the $R-R$ fields and are examples of D-branes, the second time we have come across them in the reading group so far and we will see much more of them later.

10 Week 10: 11/4/2017

We encountered T-duality this week and saw that if one of the spacetime dimensions was compact, in this example a circle with radius R, then closed strings could wind around it. This lead to a string spectrum with mass shell condition

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2),$$

here we have two new kinds of states; modes with quantised momentum $\frac{n}{R}$ and winding modes where the integer w is the winding number of the string around the circle. There was an interesting duality where closed strings on a circle of radius R are "dual" to closed strings on a circle of $\frac{\alpha'}{R}$ with the winding and quantised momentum modes swapped. In the case of the open string they don't wind so there are only the quantised momentum modes. However, in the $R \to 0$ limit these modes will disappear and the open string is only free to move in 9 dimensional spacetime. If we took more

[1]To see this consider that to get interations we need a vertex operator and by the state operator correspondence we would get this from considering the ψ_l, ψ_r oscillators. However, it is the F fields that appear in the expansion $\psi_l \psi_r$, see [1] for more details.

care and looked at the Fourier modes for the open string we would see that the spacetime parity transformation,

$$T : x^9_L(\tau+\sigma) \to x^9_L(\tau+\sigma), \quad x^9_R(\tau-\sigma) \to -x^9_R(\tau-\sigma),$$

which takes $R \to \frac{\alpha'}{R}$, swaps Neumann boundary conditions for Dirichelet boundary conditions in the x^9 direction. This has the effect of pinning the endpoints of the string so that they are only free to move in the remaining 8 spatial directions, 9 spacetime directions. The difference in position between these endpoints can then be used to define a "winding number" for the open string. Hopefully I will have time to flesh this out more at some point. Also note that considering the fermion sector if

$$T : \psi^9_r \to -\psi^9_r$$

then this will swap the sign of Γ^9 and hence the sign in the GSO projection so T-duality will take a Type IIA theory to a Type IIB theory and vice versa. An important caveat is that this interchanging of the GSO projection will only happen if we swap the sign of an odd number of terms, in other words if we T-dualise an odd number of dimensions we will swap the theory and if we T-dualise an even number of dimensions we will keep the same theory. The objects that the open string ends get pinned to are another appearance of D-branes. However, it is not yet clear to me how we are supposed to see that these are the same objects that carry the $R-R$ charge. Hopefully this will become apparent in the near future. Presumably there are constraints on how many dimensions we can T-dualise in related to the objects that can have $R-R$ charge. This is something that I need to clarify.

11 Week 11: 21/4/2017

We discussed the first half of chapter 6 in [1]. In particular we saw the specific values of p for which we have Dp-branes in type IIA and type IIB string theories. The particular D-Branes that we see in each string theory suggests that there must be different properties of the open string sector in the two theories. For example the lack of a $D8$-brane in type IIB suggests that there can't be open strings with free ends in type IIA as there T-duality in one dimension would transform a free open string to one pinned to a $D8$-brane. This requires more thought. These Dp-branes are claimed to be the carriers of the $R-R$ charge that we encountered earlier, Thus using the electromagnetic duality of the $R-R$ field strengths we can consider some Dp-branes to be electromagnetically dual to one another. An outline of these dualities is included in table 1 and table 2 for type IIA and type IIB respectively. We can see that the $D3$-brane in type IIB is self-dual and that for $p=-1$

Table 1: Electromagnetically Dual Dp-Branes and their Field strengths in type IIA string theory. The $D8$-brane is non-dynamical as the equations of motion for the 10-form field strength are trivially satisfied.

D-Brane	$D0$	$D2$	$D4$	$D6$	$D8$
$R-R$ field strength	2-form	4-form	6-form	8-form	10-form
Dual $R-R$ field strength	8-form	6-form	4-form	2-form	0-form
Dual D-Brane	$D6$	$D4$	$D2$	$D0$	N/A

there is a fully localised, in space and time, D-instanton which is dual to the $D7$-brane. We also

Table 2: Electromagnetically Dual Dp-Branes and their Field strengths in type IIB string theory.

D-Brane	D-instanton, p=-1	$D1$	$D3$	$D5$	$D7$	$D9$
$R-R$ field strength	1-form	3-form	5-form	7-form	9-form	N/A
Dual $R-R$ field strength	9-form	7-form	5-form	3-form	1-form	N/A
Dual D-Brane	$D7$	$D5$	$D3$	$D1$	D-instanton	N/A

have a non-dynamical, in the sense that the $R-R$ field strength has a trivial field equation as there are no 11-forms in 10 dimensions, $D8$-brane in type IIA and a fully space-time filling $D9$ brane in type IIB which does not couple to an $R-R$ charge. This puzzles me slightly based on the rational that the D-branes are the objects which are both required by T-duality and to have something to carry the $R-R$ charge. I need to think more about this.

As the ends of an open string carry $U(n)$ Chan-Paton charges fields charged under this will see the D-branes that the strings end on as the charge is restricted to living on the D-branes worldvolume. If we compactify some dimensions, say the x^9 direction as in [1] and above, and then include a background field in these directions which breaks the $U(n)$ symmetry down to the subgroup which preserves this background field. The example given in [1] is to take the background field

$$A_\mu = \delta_{\mu 9} \begin{pmatrix} \frac{\theta_1}{2\pi R} & & 0 \\ & \ddots & \\ 0 & & \frac{\theta_n}{2\pi R} \end{pmatrix}. \tag{11.1}$$

If all of the θ_i are distinct then this background field breaks the $U(n)$ to $U(1)^n$ and if k of the θ_i are the same then it will break the $U(n)$ to $U(k) \times U(1)^{n-k}$, so we can get both abelian and non-abelian gauge groups. In fact we can go further and see that locally A is pure gauge. This is not true globally as the function used to trivialise A is not single valued and in fact picks up a Wilson line factor,

$$\begin{aligned} W &= \exp\Big(i\int_0^{2\pi R} dx^9 A_9\Big), \\ &= \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix}, \end{aligned}$$

upon going around the compact direction. The $|k;ij\rangle$ Chan-Paton wave function will has the i'th state at one end of the string which will pick up a $e^{-i\theta_i x^9/2\pi R}$ factor under $x^9 \to x^9 + 2\pi R$, and the j'th state at the other end which picks up a factor of $e^{i\theta_j x^9/2\pi R}$ under the same translation. This difference is due to one end, the i'th state, transforming in the anti-fundamental while the other end, the j'th state, transforms in the fundamental. This results in the momentum of the state being

$$p^9_{ij} = \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R},$$

This leads to the difference in the endpoint locations of the string being

$$x'^9(\tau,\pi) - x'^9(\tau,0) = (2\pi n + \theta_j - \theta_i)\frac{\alpha'}{R},$$

where the x' are the T-dualised coordinates. From this we can say that one end point, the one carrying the i'th state, is at $\frac{\alpha'\theta_i}{R}$ and the other is at $\frac{\alpha'\theta_j}{R}$ and thus

$$x_i'^9 = \frac{\alpha'\theta_i}{R} = 2\pi\alpha'(A_9)_{ii}. \tag{11.2}$$

This gives the position of n hyperplanes, one for each value of i, and these are the D-branes. Note that so far we have $D8$-branes as we have T-dualised one dimension but by repeating the process with more dimensions we could arrive at any of the D-branes that we saw earlier. The p_{ij}^9 appear in the mass shell constraint for the string states and at level one we can get massless states when $i = j$ so the string has both ends attached to the same D-brane.The states in this case will be given by

$$\alpha_{-1}^M|k;ij\rangle \leftrightarrow V = \partial_{||}x^M$$

where M runs over all the space-time dimensions and the arrow signifies the state operator correspondence. The space-time dimensions can be split into two cases; those parallel to the Dp-brane, $\mu = 0, \ldots, p$, and those transverse to the Dp-brane, $m = p+1, \ldots, 9$. The states in directions parallel to the Dp-brane transform as a gauge field, A_μ under $SO(p,1)$ and the states perpendicular give rise to scalar fields Φ^m. Normally, [1, 7], we say that the gauge field describes the shape of the Dp-brane as a soliton and the scalar fields describe the Dp-brane as embedded in space-time. We can also interpret the gauge field and scalar fields in terms of a gauge theory on the Dp-brane. In fact as we will hopefully see more of soon we can stack Dp-branes to construct a gauge theory on the branes. The scalar fields come from breaking the Chan-Paton gauge theory down to a subgroup and are thus refered to as Higgs fields. An interesting aside, mentioned by Lennart, is to note that an open string ending on a single $D3$-brane will look like the Dirac monopole and the $U(1)$ gauge field and the Higgs field will satisfy the monopole equations. We should see more examples of construction gauge theories from stacks of D-branes soon.

12 Weeks 12 and 13: 28/4/2017 and 5/5/2017

We went through the derivation of the Born-Infeld action as an open string amplitude. As this is quite an involved computation, and we did not check all the minute details, it will not presented here as we followed the discussion in [1].

13 Weeks 14 and 15: 19/5/2017 and 26/5/2017

For week 14 we were looking at the derivation of the DBI action from the Born-Infeld action and some examples of the DBI action. Some more details of what we discussed may be added in the future. As for week 15, we discussed the link between DBI actions and dimensional reductions of super Yang-Mills.

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