# Notes on rational maps and Popov vortices

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This is a short note collecting everything that I can find about rational maps in one complex dimension to help me keep in touch with the details. The aim is to understand the properties of rational maps which are important to the study of Popov vortices.

## 1 Basics

For my purposes a rational map is a holomorphic function from $\mathbb{C}P^1$ to itself they are given by the ratio of two co-prime polynomials. Viewing $\mathbb{C}P^1$ as $\mathbb{C}\cup\{\infty\}$ we can express $R$ as

$$R(z)=\frac{P(z)}{Q(z)}=\frac{a_0+a_1z+a_2z^2+\cdots+a_nz^n}{b_0+b_1z+b_2z^2+\cdots+b_mz^m}, \tag{1.1}$$

with the condition that $a_n, b_m\neq 0$ as other wise the polynomials would be of a lower degree than claimed. We will also want to impose that one of $a_0, b_0$ is no zero, if they were both zero then the two polynomials would have a common zero at $z=0$ and thus would not be co-prime. We also need to mention that both $P$ and $Q$ are not the zero polynomial. If $P$ is the zero polynomial then $R$ is the constant function zero and if $Q$ is the zero polynomial then $R$ is the constant function $\infty$. A rational map has a degree defined in terms of the degrees of the polynomials

$$\deg(R)=\max\{\deg(P),\deg(Q)\}. \tag{1.2}$$

We are primarily interested in rational maps where $\deg(R)=\deg(P)=\deg(Q)=n$ but as the properties of the rational map depend on the relative degrees of the polynomial we shall not restrict ourselves to this case just yet.

## 2 Properties

The next thing of interest will be to see what we can say about the zeros, poles, fixed points and critical points of $R$. We will be interested in the behaviour of $R$ at infinity so it is useful to have the following expression, [3],

$$R(z)=z^{n-m}\cdot\frac{\frac{a_0}{z^n}+\frac{a_1}{z^{n-1}}+\cdots+a_n}{\frac{b_0}{z^m}+\frac{b_1}{z^{m-1}}+\cdots+b_m} \tag{2.1}$$

### 2.1 Zeros and poles of $R$

There are three cases to consider here:

$m=n$. In this case all the zeros and pole lie in $\mathbb{C}$ and there are $n=\deg(R)$ of each. It can be shown using (2.1) that

$$\lim_{z\to\infty}R(z)=\frac{a_n}{b_m}=\frac{a_n}{b_n}. \tag{2.2}$$

$n>m$. There are $n$ zeros and $m$ poles in $\mathbb{C}$ and $R(\infty)=\infty$ so $\infty$ is a fixed point of $R$. We can calculate the order of the pole to be $n-m$ using (2.1). We could also do this by looking at the zeros of $\frac{1}{R(1/z)}$. Now $\deg(R)=n$ and there are $n$ zeros and poles in $\mathbb{C}P^1$.

$n < m$. Again there are $n$ zeros and $m$ poles in $\mathbb{C}$ but now $R(\infty) = 0$ so $\infty$ is a zero of the rational map of order $m - n$, again this is found using (2.1). In this case $\deg(R) = m$ and there are $m$ zeros and poles in $\mathbb{C}P^1$.

Note that by considering the zeros of $R(z) - w$ for $w \in \mathbb{C}P^1$ we can see that $R$ is a $d$-fold map of $\mathbb{C}P^1$ to $\mathbb{C}P^1$.

### 2.2 Fixed points of $R$

In the previous subsection we encountered the fact that for $n > m$ $R$ has a fixed point at infinity, in fact this is the only time when infinity is a fixed point as we saw by considering teh behaviour of $R$ at infinity in the three cases above. It is of interest to know how many fixed points a generic $R$ will have. To count them we proceed following [2]; if $w \in \mathbb{C}$ is a fixed point then $Q(w) \neq 0$, other wise $R(w) = \infty$, so we have

$$P(w) = wQ(w). \tag{2.3}$$

We can also see the converse that if (2.3) holds then $Q(w) \neq 0$ as otherwise $P$ and $Q$ would have a common root at $w$ and would not be co-prime so $w$ is a fixed point of $R$. This means that the fixed points are the solutions of (2.3). It can be shown, Theorem 2.6.2 in [2], that if $R$ is a rational map and $g$ is a mobius transformation then $gRg^{-1}$ has the same number of fixed points at $g(w)$ as $R$ has at $w$. By the number of fixed points of $R$ at $w$ we mean the number of zeros of $R(z) - z$ at $z = w$. In practice this invariance of the number of fixed points under conjugation is how the number of fixed points at infinity is counted.

The result we want on the number of critical points of a rational map is

**Theorem 2.1** (Theorem 2.6.3 in [2])**.** *If $d \geq 1$, a rational map of degree $d$ has precisely $d + 1$ fixed points in $\mathbb{C}P^1$.*

The proof is replicated here for ease of reference.

*Proof.* As any rational map is conjugate to one which does not fix infinity we shall assume that $R$ does not fix infinity. Now if $w \in \mathbb{C}$ is a fixed point of $R$ then the number of zeros of $R(z) - z$ at $w$ is the same as the number of zeros of $P(z) - zQ(z)$ at $w$, this is the number of solutions to (2.3) in $\mathbb{C}$. As $R$ does not fix infinity we have that

$$n \leq m = \deg(R). \tag{2.4}$$

This means that

$$P(z) - zQ(z) \tag{2.5}$$

has degree $\deg(R) + 1$. □

Each fixed point, $w \in \mathbb{C}P^1$, has a complex number called the multiplier, $M(R, w)$ attached to it which is given by

$$M(R, w) = \begin{cases} R'(w) \text{ if } w \neq \infty, \\ \dfrac{1}{R'(\infty)} \text{ if } w = \infty. \end{cases} \tag{2.6}$$

The multiplier is conjugation invariant.

### 2.3 Critical points of $R$

The critical points of $R$ are the points, $z$, such that $R$ fails to be injective in a neighbourhood around $z$. A critical value is the image of a critical point. Now if $R$ has degree $d$ then for a non-critical value $w$, $R^{-1}(w)$ is a set consisting of $d$ distinct points, $\{z_1, \ldots, z_d\}$. As the $z_j$ are not critical points there is a neighbourhood around each such that $R$ is injective. Before we can calculate the number of fixed points we need to know about the valency of $R$ at a point $z$.

#### 2.3.1 Valency of a rational map

We take as our definition of the valency of an analytic function $f$ at a point $z_0$, $v_f(z_0)$, the order of the zero of $f(z) - f(z_0)$ at $z_0$. In other words it is the $k$ such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)^k} = C \tag{2.7}$$

for $C$ a finite non-zero constant. A related idea is how much $f$ fails to be injective, in fact if $f$ is injective near $z_0$ then $v_f(z_0) = 1$. Another way to think about it is that $v_f(z_0)$ is the number of solutions to $f(z) = f(z_0)$ at $z_0$.

The critical points of a, non-constant, rational map $R$ will thus be the points $z$ such that $v_R(z) > 1$. In fact we can use the fact that $R$ is a $d$-fold map to express the degree of $R$ in terms of the valence of points in the pre-image of $w \in \mathbb{C}P^1$, independent of which $w$ we pick. This relation takes the form

$$\sum_{z \in R^{-1}(w)} v_R(z) = \deg(R). \tag{2.8}$$

Now $R$ is injective when $R'$ has neither a pole nor a zero so there are only a finite set of critical values of $R$ and thus $v_R(z) = 1$ for all but finitely many $z \in \mathbb{C}P^1$. This means that

$$\sum_{z \in \mathbb{C}P^1} (v_R(z) - 1) < \infty. \tag{2.9}$$

#### 2.3.2 Counting critical points

The sum in (2.9) gives a measure of the number of multiple roots of $R$. Its value is the subject of the Riemann-Hurwitz theorem.

**Theorem 2.2** (Riemann-Hurwitz, Theorem 2.1.7 in [2])**.** *For any non-constant Rational map $R$,*

$$\sum_{z \in \mathbb{C}P^1} (v_R(z) - 1) = 2deg(R) - 2. \tag{2.10}$$

The terms in the sum are only non-zero when $z$ is a critical point so the sum can be used to estimate the number of critical points of a rational map. A corollary of Theorem 2.2 is then:

**Corollary 2.3.** *A rational map of degree $d$ has at most $2d - 2$ critical points in $\mathbb{C}P^1$.*

*Proof.* The proof is fairly straight forward, the number of terms in the sum on the left hand side of (2.10) is the number of critical points, $N$, and as $v_R(z) \geq 1$ for all $z \in \mathbb{C}P^1$ we have that

$$2d - 2 = \sum_{z \in \mathbb{C}P^1} (v_R(z) - 1) \geq N. \tag{2.11}$$

□

If we think of $v_R(z) - 1$ as the multiplicity of a critical point $z$ then counted with multiplicity there are exactly $2d - 2$ critical points.

I will not prove Theorem 2.2 here but a proof is given in [2] and in many other good complex geometry texts. Notice that there is no restriction here on whether $\infty$ is a critical point or not. This is because if infinity is a critical point we can conjugate with a mobius transformation and work with a rational map where $\infty$ is not a critical point.

## 3 Application to Popov vortices

Here we change tack and consider Popov vortices on $\mathbb{C}P^1$ and follow the discussion in [4] to see how they are related to rational maps. To see that Popov vortices correspond to rational maps we will proceed in two stages; stage one is showing that a Popov vortex corresponds to a constant curvature 1 metric on $\mathbb{C}P^1$ with finitely many conical singularities. While steage two will be to invoke Theorem 1.3 in [5] which says that all such constant curvature 1 metrics with finitely many conical singularities correspond to pulling back the standard metric on $\mathbb{C}P^1$ by a rational map.

Stage one is the part that will require some work. Note that the Popov vortex equations on the 2-sphere of radius 1,

$$\partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0, \qquad F = da = -(1 - |\phi|^2)\mathcal{R}. \tag{3.1}$$

can be shown to be imply the equation

$$\partial_z \partial_{\bar{z}} u = \frac{2}{(1 + |z|^2)^2}(1 - e^u), \tag{3.2}$$

where $e^u = |\phi|^2$. We can solve equation (3.2) explicitly in the following way. Consider a generic metric with Gauss curvature $K$,

$$ds^2 = \Omega(z, \bar{z}) dz d\bar{z}, \tag{3.3}$$

on a Riemann surface S. We know that

$$K = -\frac{2}{\Omega}\partial_z \partial_{\bar{z}} (\ln \Omega). \tag{3.4}$$

The conformally related metric

$$ds^2 = e^{v(z,\bar{z})}\Omega(z, \bar{z}) dz d\bar{z}, \tag{3.5}$$

with Gauss curvature $K'$ we have that

$$K' = -\frac{2}{e^v \Omega}\partial_z \partial_{\bar{z}} (v + \ln \Omega) = \frac{1}{e^v}\left(-\frac{2}{\Omega}\partial_z \partial_{\bar{z}} v + K\right). \tag{3.6}$$

If we take $K' = K = 1$ and $\Omega = \frac{4}{(1+|z|^2)^2}$, the conformal factor for the standard metric on the sphere we have that

$$\partial_z \partial_{\bar{z}} v = \frac{2}{(1+|z|^2)^2}(1 - e^v), \tag{3.7}$$

which is the same as (3.2) for $v = u$. Thus a Popov vortex defines a conformal metric with constant curvature 1. TO see that this metric has finitely many conical singularities note that the metric (3.5) degenerates when $|\phi| = 0$ and that imposing finite energy implies that $\phi$ only has finitely many zeros. By analysing (3.5) around $e^v = 0$ we can see that the metric is conical around these points. Thus we can say that a Popov vortex, on the 2-sphere of radius 1, is equivalent to a constant curvature 1 metric with finitely many conical singularities, or degeneracies in this case. Thus Theorem 1.3 in [5] can be used to tell us that a Popov vortex is equivalent to a rational map.

To see the exact relation consider the rational map $R : \mathbb{C}P^1 \to \mathbb{C}P^1$ then

$$R^* ds^2 = \frac{4R'(z)\overline{R'(z)}}{(1+|R(z)|^2)^2} dz d\bar{z} = \frac{R'(z)\overline{R'(z)}(1+|z|^2)^2}{(1+|R(z)|^2)^2} ds^2_{\mathbb{C}P^1}. \tag{3.8}$$

In other words

$$|\phi|^2 = e^u = \frac{(1+|z|^2)^2}{(1+|R(z)|^2)^2} R'(z)\overline{R'(z)}. \tag{3.9}$$

The zeros of $|\phi|$ will then be the critical points of $R$. Here is where we need to be careful, our stereographic chart does not include the south pole as that is the point we are projecting from and corresponds to the point at infinity.

Arriving at these results has not involved making any assumption about the form of the rational map $R$ so we can come to the conclusion that **all rational maps correspond to Popov vortices**! The conditions placed on $R$ in Equation (21) of [4], that $a_0, b_0, a_n, b_n \neq 0$, seem to be there just to ensure that the two kinds of singularity present can be clearly seen. Another reason for placing further restrictions on $R$ would be to avoid having a vortex outside the chart we are working in. This would be precisely what would happen if

### 3.1 Some examples

**Example 3.1.** Take $R(z) = z^k$ for $k > 1$. There are two critical points, 0 and $\infty$ each with multiplicity $k - 1$. In fact as stated in [4] The vortex solution is circularly symmetric, as $\phi$ depends only on $|z|^{k-1}$ and reflection symmetric in $|z| = 1$.

**Example 3.2.** A degree 1 rational map, otherwise known as a mobius transformation, will correspond to the zero vortex case, $N = 0$. These non-trivial solutions will have different Higgs fields but in all the cases $\phi$ will not have any zeros or winding.

## References


[1] Notes on iteration of rational functions by Erin Pearse at `http://www.math.cornell.edu/~erin/docs/iteration.pdf` accessed August 2017.

[2] A. F. Beardon. Iteration of Rational Functions: Complex Analytic Dynamical Systems, Springer, New York, 1991.

[3] PlanetMath article on the zeros and poles or rational functions at `http://planetmath.org/zerosandpolesofrationalfunction` accessed August 2017.

[4] N. S. Manton. Vortex solutions of the Popov equations. *Journal of Physics A Mathematical General*, 46(14):145402, 2013.

[5] Q. Chen, W. Wang, Y. Wu, B. Xu Conformal metrics with constant curvature one and finite conical singularities on compact Riemann surfaces. arXiv 1302.6457