Notes related to the Nahm's equations and hyperkähler quotient constructions reading group

Calum Ross

Last updated July 6, 2022

Hopefully I will tex up notes on some of the topics that we cover during our reading group following [1]. I don't mean to suggest that I will make exhaustive notes or even any notes on most of the discussions. However, I will try and include the important points that we discuss. A webpage containing the details of the reading group and links to further references is available here. Depending how quickly we progress we may also try to understand the hyperkähler quotient construction of the moduli space of solutions to Hitchin's equations following [2], I have previously made some notes on aspects of this topic based on a talk that I gave at the 2017 BIG workshop on Higgs bundles.

1 Week 1: 1/06/2022

We had a general house keeping meeting where we decided that we wanted to read [1] and understand the example of an infinite dimensional hyperkähler quotient construction coming from Nahm's equations. For the 8th, 15th, and 22nd of June I have booked room 707 for us to use.

2 Week 2: 8/06/2022

I will give some general motivation and discuss the dimensional reduction of the ASD equations to give BPS monopole, Hitchin equations, and the Nahm equations. It will roughly follow section 2 of [1].

A sketch of what we discussed is included here, if I have the time I will try to flesh it out in more detail and address some of the comments that I have added.

General Motivation/ Why Quotients

The idea of why we care about quotients is easiest to see in the finite dimensional case where we have a Lie group acting on a manifold $G \circlearrowleft M$, if this action is "nice" then the "unique" configurations are given by the quotient M/G. The natural physics setting for this is gauge theory[1] The ingredients are

- Manifold M,
- Lie group G (assumed compact),
- Principal G-bundle $P \to M$ (and its associated vector bundles $E \to M$ usually the adjoint representation),
- gauge field are connections on P, the space of gauge fields, $\mathcal{A}$, is an affine space modelled on the $\mathfrak{g}$ valued 1-forms (often written $\Omega^1_P(\mathfrak{g})$),

[1] It also shows up in lots of other cases, e.g. translation or rotation symmetries in physical theories. However, these are all finite dimensional examples. The natural source of infinite dimensional group actions on infinite dimensional spaces is in gauge theory.

- Acting on $\mathcal{A}$ is the group of gauge transformations $\mathcal{G}$, these are sections of the Adjoint bundle, when P is trivial
$$\mathcal{G} = \{g : M \to G\} = C^\infty(M, G), \tag{2.1}$$
- $\mathcal{G} \circlearrowleft \mathcal{A}$ as
$$A \mapsto A^g = gAg^{-1} + gdg^{-1}. \tag{2.2}$$

The connections of interest will solve an equation coming from an energy functional: e.g. when $G = SU(2)$ and $M = \mathbb{R}^4$ take
$$E = -\frac{1}{8\pi^2}\int \mathrm{Tr}\left(F \wedge \star F\right), \tag{2.3}$$
The critical points of this energy are connections A such that $d_A \star F_A = 0$, while the absolute minima are called instantons and satisfy the ASD equations $F_A = -\star F_A$ with energy $E = k \in Z$. If A is an instanton then A^g is also an instanton with the same energy k. (If A satisfies any boundary conditions then they may only be preserved by $g \in \mathcal{G}_0$ e.g. $g \to \mathbb{I}$ asymptotically[2]). Then "physical" configurations are given by
$$\{A \in \mathcal{A} | d_A \star F_A = 0\}/\mathcal{G}_0, \tag{2.4}$$
and "physical" instantons with energy k are in the moduli space
$$\mathcal{M}_k = \{A \in \mathcal{A} | F_A = - \star F_A\}/\mathcal{G}_0. \tag{2.5}$$
The question that we want to answer in this reading group is "How are these quotients computed?" If everything was finite dimensional there is a standard theory that could be applied however, $\dim\mathcal{A}$ = "∞" = $\dim\mathcal{G}_0$, so how do we know that $\mathcal{M}_k$ is even a manifold?

This can be proved, however it is done in a case by case basis. e.g.

- YM on a Riemann surface [3],
- Hitchin equations on a Riemann surface [2],
- Moduli space of magnetic monopoles on $\mathbb{R}^3$ [4] (this case is really done by making use of the Nahm transform and computing the moduli space of solutions to Nahm's equations which is exactly the problem that we will study in this reading group).

Instantons all the way down

Instantons are solitons in four dimensions by imposing symmetries on them we get other interesting objects, see Table 1 for what happens to translationally invariant instantons.

Manifold	$\mathbb{R}^4$	$\mathbb{R}^3$	$\mathbb{R}^2$	$\mathbb{R}^1$	$\mathbb{R}^0$
Coordinates	(x^0, x^1, x^2, x^3)	(x^1, x^2, x^3)	(x^0, x^1)	(x^0)	None
Invariances	None	x^0	x^2, x^3	x^1, x^2, x^3	x^0, x^1, x^2, x^3
Fields	A	$A, \Phi = A_0$	$A, \Phi = (A_3 - iA_2)\frac{dz}{2}$	$A_0, \vec{A}$	A_i
ASD eqs	$F_A = - \star F_A$	$F_A = \star d_A \Phi$	$\bar{\partial}_A \Phi = F_A + [\Phi, \Phi^*]$	Nahm equations	ADHM equations

Table 1: Instantons and their descendents.

These different dimensional instantons are not independent, some of them are related to each other through the Nahm transform. I may add more on this later depending on if we discuss the Nahm transform more in this reading group.

[2]or at least g is asymptotically in the same connected component as the identity.

Dimensional reductions

In the session I only discussed the Nahm case however, I have notes on the other cases that I will try to tex up. The general idea is to consider a principal G-bundle over $\mathbb{R}^4$ with (A)SD connections (instantons) on it. If we have a symmetry group on $\mathbb{R}^4$, e.g. Γ a subgroup of the translations on $\mathbb{R}^4$, then it is natural to ask what happens if an instanton A is invariant under Γ. In other words can we interpret A as a connection on a bundle over $\mathbb{R}^4/\Gamma$ plus some extra fields. This extends beyond the case of $M = \mathbb{R}^4$ and for more general symmetries than translations.

Instantons

Let $M = \mathbb{R}^4$ and take the trivial G-bundle $P = M \times G$. Then a connection A and its curvature $F = d_A A$ are given by

$$A = \sum_{i=0}^{3} A_i dx^i, \tag{2.6}$$

$$F = \sum_{i,j=0}^{3} \frac{1}{2} F_{ij} dx^i \wedge dx^j, \tag{2.7}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{2.8}$$

The ASD equations are

$$\begin{aligned} F_{01} + F_{23} &= 0, \\ F_{02} + F_{31} &= 0, \\ F_{03} + F_{12} &= 0, \end{aligned} \tag{2.9}$$

which are non-linear PDEs in the connection components A_i.

Since P is trivial the group of gauge transofrmations is given by G valued functions

$$\mathcal{G} = \{g : \mathbb{R}^4 \to G\} \tag{2.10}$$

and acts on connections via the adjoint action

$$g \cdot A = \mathrm{Ad}_g A - (dg)\, g^{-1} = gAg^{-1} + gd\left(g^{-1}\right). \tag{2.11}$$

The derivative of g is an element of the tangent space $\partial_i g \in T_{g(x)}G$ and is thus defined by using the right action $(\partial_i g(x))\, g^{-1}(x) = dR_{g^1(x)}\left(\partial_i g(x)\right) \in T_1 G = \mathfrak{g}$.

The energy functional for Instantons is the Yang-Mills functional

$$YM : A \mapsto -\frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(F_A \wedge \star F_A\right) \tag{2.12}$$

which is an integer for an instanton, the second Chern number of P.

BPS monopoles

If the symmetry group Γ is translations in x^0 then instantons reduce to monopoles, in particular what are known as BPS monopoles in the physics literature[8]. Let $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ where we think of $x^0 = t$ as the coordinate on the $\mathbb{R}$ factor. Then a four dimensional connection $\tilde{A}$ reduces to

$$\tilde{A} = \Phi dt + \sum_{i=1}^{3} A_i dx^i = \Phi dt + A, \tag{2.13}$$

with $\Phi, A_i : \mathbb{R}^3 \to \mathfrak{g}$. Here A is a connection on a G-bundle over $\mathbb{R}^3$, trivial if the bundle over $\mathbb{R}^4$ was trivial, and Φ is a section of the adjoint bundle known as the Higgs field. The ASD equations become

$$\star\, d_A \Phi = F_A \tag{2.14}$$

which are known as the Bogomol'nyi equations. These equations make sense on any three-manifold and have been widely studied, mostly for $G = SU(n)$, but also for the other classical groups.

Hitchin System

We can also dimensionally a second time to get what are known as the Hitchin equations introduced in [2]. Write $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and take Γ to be translations in the second factor. A connection $\tilde{A}$ on $\mathbb{R}^4$ then becomes

$$\tilde{A} = A_0 dx^0 + A_1 dx^1 + \phi_2 dx^2 + \phi_3 dx^3 = A + +\phi_2 dx^2 + \phi_3 dx^3, \tag{2.15}$$

again we have $A_i, \phi_i : \mathbb{R}^2 \to \mathfrak{g}$. Thus get a connection A on the G-bundle over $\mathbb{R}^2$ but now we interpret the Higgs field as

$$\Phi = \frac{1}{2} \left(\phi_3 - i\phi_2\right) dz \in \Omega^{1,0}_{\mathbb{C}} \left(\mathrm{ad}\,(P) \otimes \mathbb{C}\right), \tag{2.16}$$

with $z = x^0 + ix^1$ the complex coordinate on $\mathbb{R}^2$. The ASD equations then become

$$F_A + [\Phi, \Phi^*] = 0, \tag{2.17}$$

$$\bar{\partial}_A \Phi = 0. \tag{2.18}$$

These equations are conformally invariant so make sense on any compact Riemann surface σ.

This is not the only way to reduce instantons to two dimensions. If $G = SU(2)$ we can use the conformal equivalence $\mathbb{R}^4 \simeq H^2 \times S^2$ then if we impose an $SO(3)$ invariance we reduce the ASD equations to PDEs on two dimensional hyperbolic space H^2. These are known as the vortex equations in the Abelian-Higgs model [8] and from the physics point of view they are of more interest than the Hitchin equations due to their relationship with effective models of superconductors and superfluids.

Nahm equations

Consider $\mathbb{R}^4 = M = \mathbb{R} \times \mathbb{R}^3$ and impose invariance on the $\mathbb{R}^3$. An instanton on $\mathbb{R}^4$, and teh associated bundle P descend to $\mathbb{R}$ where the instanton becomes

$$A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 = A + A_i dx^i. \tag{2.19}$$

Since $A_i : \mathbb{R} \to \mathfrak{g}$ combine them into a "Higgs field"

$$\Phi = A_1 i + A_2 j + A_3 k : \mathbb{R} \to \mathfrak{g} + \mathrm{Im}\mathbb{H}. \tag{2.20}$$

Often the notation $T_0, \vec{T}$ is used rather than A_0, A_i but I will try to stick with the notation used in [1].

The ASD equations reduce to

$$\star\, d_A\Phi + \frac{1}{2}[\Phi, \Phi] = 0, \tag{2.21}$$

or

$$\dot{A}_i + [A_0, A_i] + \frac{1}{2}\varepsilon_{ijk}[A_j, A_k] = 0, \tag{2.22}$$

where we sum over j, k and there is an equation for each $i = 1, 2, 3$. Checking this dimensional reduction explicitly is left as an exercise to the especially interested reader.

The action of the group of gauge transformations preserves Eq. (2.22), and $g \in \mathcal{G}$ acts on an element of $\mathcal{A}$ as

$$g \cdot (A_0, \vec{A}) = \left(\mathrm{Ad}_g A_0 - \dot{g}g^{-1}, \mathrm{Ad}_g \vec{A}\right), \tag{2.23}$$

N.B. this matches ths earlier definition of a gauge transformation since $dgg^{-1} = -gdg^{-1}$.

When G is a unitary group e.g. $U(n)$ it is useful to think of the associated rank n vector bundle $E \to I = [0, 1]$. We have swapped to considering Nahm over a compact interval[3] as that is the most commonly studied case in the literature. Following on from Jakob's question this seems to be because monopoles on $\mathbb{R}^3$ correspond to Nahm equations on I, the specifics follow from looking at the details of the Nahm transform if we cover this later I will add the details. Then

$$A_0 \to \nabla : \Omega^0_I(E) \to \Omega^1_I(E) \tag{2.24}$$

becomes a connection on a vector bundle and the A_i are skew adjoint sections of End (E), in trivialisation they are just skew adjoint matrices. If the coordinate on I is t then Nahm's equations become

$$\nabla_t A_i + \frac{1}{2}\varepsilon_{ijk}[A_j, A_k] = 0. \tag{2.25}$$

The group of gauge transformations are now the automorphisms of E which preserve the hermitian metric, $\mathcal{G} = \mathrm{Aut}_0\,(E)$, $g \in \mathcal{G}$ acts by conjugation on ∇_t and on $\vec{A}$.

Example 2.1. Let $G = SU(2)$ and take its basis to be $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{su}(2)$ such that $[\sigma_i, \sigma_j] = \varepsilon_{ijk}\sigma_k$ e.g. $\sigma_i = -\frac{i}{2}\tau_i$ for τ_i the Pauli matrices.

Then $\left(0, \frac{\vec{\sigma}}{t}\right)$ is a solution to Eq. (2.25) with a first order pole at $t = 0$. More generally, there are solutions in terms of Jacobi elliptic functions given in terms of an elliptic modulus $0 \le k \le 1$ and a parameter $0 \le D \le 2K(k)$, for $K(k)$ the complete elliptic integral of the second kind. Taking

$$(A_0, A_1, A_2, A_3) = (0, f_1\sigma_1, f_2\sigma_2, f_3\sigma_3) \tag{2.26}$$

Nahm's equations reduce to the Euler-Poinsot equations for a spinning top which are solved by

$$f_1(t) = \frac{D\mathrm{cn}_k\,(Dt)}{\mathrm{sn}_k\,(Dt)}, \tag{2.27}$$

$$f_2(t) = \frac{D\mathrm{dn}_k\,(Dt)}{\mathrm{sn}_k\,(Dt)}, \tag{2.28}$$

$$f_3(t) = \frac{D}{\mathrm{sn}_k\,(Dt)}. \tag{2.29}$$

[3] Here the interval is $[0, 1]$ which matches the conventions in section 2 of [1], however in many places when we carry out the Nahm transform of a monopole we get the interval to be $[-1, 1]$.

These solutions also have first order poles at $t = 0$ and for $D \to 0$ these reduce to the initial examples. When $D = 2K(k)$ there is also a pole at $t = 1$, and when $k \to 0$ these become trigonometric

$$f_1(t) = \frac{D\cos(Dt)}{\sin(Dt)}, \tag{2.30}$$

$$f_2(t) = f_3(t) = \frac{D}{\sin(Dt)}. \tag{2.31}$$

Acting on these solutions with the group of gauge transformations,

$$\mathcal{G}_0 = \{g : I \to G | g(0) = \mathbb{I}\} \tag{2.32}$$

and by an $SO(3)$ action known as hyperkähler rotation[4] gives all the $SU(2)$ solutions to Nahm's equations with these poles and residues.

3 Week 3: 15/06/2022

Jaime gave a run through of hyperkähler geometry and the various quotient constructions

Hyperkähler manifolds

Definition 3.1. *A Hyperkähler (HK) manifold is a smooth manifold M with a Riemannian metric g and three complex structures I, J, K which are Kähler with respect to g and satisfy $IJK = -1$.*

Remark 3.2. *I, J, K behave like the quaternions i, j, k and we have that T_pM is a quaternionic space $\forall p \in M$. We also have that the dimension of M is a multiple of 4, $dim M = 4n$.*

The three Kähler structures give us three symplectic structures $\omega_I, \omega_J, \omega_K$. By singling out one of the complex structures we can turn the other two into a holomorphic symplectic structure $\omega_{\mathbb{C}}$ which is holomorphic symplectic with respect to (M, g, I, ω_I). This means that HK manifolds are holomorphic symplectic manifolds.

Remark 3.3. *They can be defined as Riemannian manifolds with holonomoy in the compact symplectic group $Sp(n) \subset GL(4n, \mathbb{R})$. Note that the holomorphic symplectic interpretation arises because $Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$.*

Riemannian quotients

Theorem 3.4. *Let (M, g) be a Riemannian manifold and G a compact Lie group. If $G \circlearrowleft M$ freely through isometries then M/G inherits a unique Riemannian metric $\bar{g}$ such that*

$$\bar{g}(X, Y) \circ \pi = g(X^*, Y^*) \tag{3.1}$$

$\forall X, Y \in \Gamma(T(M/G))$ and X^*, Y^* the horizontal lift to $\Gamma(TM)$. Moreover, if (M, g) is complete the $(M/g, \bar{g})$ is complete.

The idea is to build on this type of quotients by considering group actions which preserve other structures on M.

[4] this is the action $q \in SP(1) \subset \mathbb{H}^*$ $A \mapsto q \cdot A = qAq^{-1}$ which is an $SO(3)$ action since q and $-q$ act the same on A.

Kähler quotients

Consider a Kähler manifold (M, g, ω, I) and a compact Lie group $G \circlearrowleft M$ freely[5] preserving both the metric and the Kähler structure. The example to have in mind is $U(1) \circlearrowleft \mathbb{C}^n$ where the action is

$$e^{i\theta} \cdot (z_1, \dots, z_n) \mapsto (e^{i\theta} z_1, \dots e^{i\theta} z_n). \tag{3.2}$$

Be aware that this action is not free, the origin is a fixed point.

As a first step consider symplectic reduction. Let $x \in \mathfrak{g}$ and $x^{\#} \in \Gamma(TM)$ be the associated infinitesimal action. Saying that the G action preserves the symplectic sturcture means that

$$0 = \mathcal{L}_{x^{\#}} \omega = i_{x^{\#}} d\omega + d\left(i_{x^{\#}} \omega\right) = \left(i_{x^{\#}} \omega\right). \tag{3.3}$$

Thus $i_{x^{\#}} \omega$ is a closed one-form. If it is exact then

$$i_{x^{\#}} \omega = d\mu_x \tag{3.4}$$

and the action is called Hamiltonian, μ_x is called a Hamiltonian function, and $x^{\#}$ a Hamiltonian vector field. Combining all the generators of $\mathfrak{g}$ gives a map

$$\mu : M \to \mathfrak{g}^*, \qquad \mu : p \mapsto (x \mapsto \mu_p(x)). \tag{3.5}$$

If this map is G equivariant with respect to the coadjoint action on $\mathfrak{g}^*$ then μ is called a moment map.

Going back to our example of $U(1) \circlearrowleft \mathbb{C}^n$, the symplectic form is $\omega = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ and the vector field generating the infinitesimal action is

$$x^{\#} = X = i \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \tag{3.6}$$

Computing the moment map gives

$$d\mu = i_X \omega = -\sum_{i=1}^{n} \left(z_i d\bar{z}_i + \bar{z}_i dz_i\right) = d\left(-\frac{1}{2}\|z\|^2 + c\right), \tag{3.7}$$

where c is an arbitrary constant. Typically $c = 1$ is taken so that $\mu^{-1}(0)$ gives the sphere with $\|z\|^2 = 2$ rather than just a point. This subtlety is due to the action having a fixed point.

If the action is free then $\mu^{-1}(0)$ is a smooth submanifold with an inclusion $\iota : \mu^{-1}(0) \hookrightarrow M$. The pullback $\iota^* \omega$ on $\mu^{-1}(0)$ is in general degenerate. However, the symplectic complement of the tangent space $T_p \mu^{-1}(0)$ is the same as the tangent space to the orbit through the point p $T_p (G \cdot p)$. Thus the degenerate directions come from the directions along the orbit and quotienting by G removes the degeneracy. Hence

$$M//G = \mu^{-1}(0)/G \tag{3.8}$$

inherits a symplectic form $\bar{\omega}$ defined through $\pi^* \bar{\omega} = \iota^* \omega$ where $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$. Thus $(M//G, \bar{\omega})$ is a symplectic manifold of dimension $\mathrm{dim} M//G = \mathrm{dim} M - 2\mathrm{dim} G$. If M has a complex structure I then this descends to a complex structure $\bar{I}$ with integrability preserved, and $\left(M//G, \bar{g}, \bar{\omega}, \bar{I}\right)$ is called the Kähler reduction.

[5] If the action has fixed points as in the example of $U(1) \circlearrowleft \mathbb{C}^n$ a quotient can still be taken if the fixed point is removed, or if we are careful about which level set we work on.

In the $U(1) \circlearrowleft \mathbb{C}^n$ example we saw that the level set $\mu^{-1}(0) = S^{2n-1} \subset \mathbb{C}^n$ and thus

$$C^n//U(1) = \mu^{-1}(0)/U(1) = S^{2n-1}/U(1) = \mathbb{C}P^{n-1}. \tag{3.9}$$

This is true for every level except the one where $\|z\|^2 = 0$, e.g. $\mu^{-1}(c)$, this is why we need the constant c for $\mu^{-1}(0)$ to be a suitable level set.

Warning, this is sketchier as it is something that we do not understand as well as the previous topics. An alternative picture of Kähler quotients is possible if the G action extends to a holomorphic action of $G_{\mathbb{C}}$. This is the group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ such that $G \subset G_{\mathbb{C}}$ is the maximum compact subgroup[6] We believe that this is unique. The action extending is equivalent to $Ix^{\#}$ being complete $\forall x \in \mathfrak{g}$ e.g. $Ix^{\#}$ gives the imaginary infinitesimal action. Then

$$M//G = M^{ss}/G_{\mathbb{C}}, \tag{3.10}$$

where the ss superscript denotes the semistable points of M, e.g. we have excluded the "bad" orbits like fixed points.

Consider $U(1)_{\mathbb{C}} \circlearrowleft \mathbb{C}^2$, then $(\mathbb{C}^n)^{ss]} = \mathbb{C}\backslash\{0\}$ since the only "bad" point is the fixed point at the origin. Then

$$\left(\mathbb{C}\backslash\{0\}\right)/\mathbb{C}^* = \mathbb{C}P^n = \mathbb{C}^n U(1). \tag{3.11}$$

HK quotients

Let (M, g, I, J, K) be a HK manifold and G a compact Lie group acting freely[7] on M preserving the HK structure. We could hope to get a triple of moment maps, one for each symplectic/ complex structure,

$$\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}. \tag{3.12}$$

e.g We need to have three Hamiltonian functions.

Then observe that $\mu_{\mathbb{C}} = \mu_J + I\mu_K : M \to \mathfrak{g}^*_{\mathbb{C}}$ is holomorphic with respect to I and $\mu^{-1}(0) \subset M$ is a Kähler submanifold of (M, g, ω_I, I). This still leaves $\mu_I : \mu^{-1}(0)_{\mathbb{C}} \to \mathfrak{g}^*$, the moment map for $G \circlearrowleft (\mu^{-1}(0), g, \omega_I, I)$ which we can use to do a Kähler reduction to find

$$M///G = \mu_{\mathbb{C}}^{-1}(0)//G = \mu_{\mathbb{C}}^{-1}(0) \cap \mu_I^{-1}(0)/G = \mu^{-1}(0)/G. \tag{3.13}$$

This is called the HK reduction as we get the same $M///G$ complete with HK structures $\bar{I}, \bar{J}, \bar{K}$ regardless of which complex structre is singled out when doing the reduction. As it inherits all three complex structures $M///G$ is a HK manifold.

This also has a holomorphic symplectic interpretation with $\omega_{\mathbb{C}} = \omega_J + I\omega_K$ a holomorphic symplectic structure on (M, g, ω_I, I) with $\mu_{\mathbb{C}}$ the holomorphic moment map of the extended $G_{\mathbb{C}}$ action on $(M, I, \omega_{\mathbb{C}})$. Then the holomorphic symplectic quotient is

$$\text{“}M//G_{\mathbb{C}}\text{”} = \mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}} = \mu^{-1}(0)/G. \tag{3.14}$$

The dimension of the quotient is

$$\dim\left(M///G\right) = \dim M - 4\dim G. \tag{3.15}$$

[6] As an example think of $\left(U(1)^n\right)_{\mathbb{C}} = (\mathbb{C}^*)^n$. Note that this $G_{\mathbb{C}}$ action will not necessarily preserve all of the structure e.g. $U(1)_{\mathbb{C}} = \mathbb{C}^* \circlearrowleft \mathbb{C}^n$ preserves the complex structure but not the metric.

[7] Again usually freely as we can deal with some "badly" behaved orbits.

There is actually a whole S^2 of complex structures rather than just three since $aI + bJ + cK$ is a complex structure when $(a, b, c) \in S^2 \subset \mathbb{R}^3$. The quotient story holds for any of these complex structures and is related to the twistor space of M.

Example 3.5 (From [5])**.** Note that this needs to be rechecked as it looks like I am secretly using a different convention to Hitchin. I believe that the details given here are self consistent but the moment maps do not agree exactly with those in [5]. In fact the differing convention is in the definition of I, J, K, I take them to extend the left action of i, j, k and Hitchin takes them to extend the right action, comparing the two then involves a conjugation which should explain the difference

Consider $U(1) \circlearrowleft V \oplus V^*$, where V is a Hermitean vector space and V^* is its dual. There is a natural hyperkähler triple on $V \oplus V^*$ given by

$$I : (\partial_z, \partial_w) \to i(\partial_z, \partial_w), \quad J : (\partial_z, \partial_w) \to (i\partial_{\bar{w}}, -i\partial_{\bar{z}}), \quad K : (\partial_z, \partial_w) \to (-\partial_{\bar{w}}, \partial_{\bar{z}}). \tag{3.16}$$

Exercise to the reader, check that $JK(\partial_z, \partial_w) = I(\partial_z, \partial_w)$. The $U(1)$ action is given by $u \in U(1)$

$$u \cdot (z, w) = (uz, u^{-1}w). \tag{3.17}$$

We can think of $V = \mathbb{C}^n$, then the $n = 2$ case gives the Eguchi-Hansen space with its familiar hyperkähler metric.

The statement in [5] is that the complex moment map for I is

$$\mu_I(z, w) = \|z\|^2 - \|w\|^2, \tag{3.18}$$

and the holomorphic symplectic moment map is

$$\mu_{\mathbb{C}}(z, w) = \mu_J(z, w) + I\mu_K(z, w) = w(z). \tag{3.19}$$

To prove this is the case consider that

$$\omega_I = \frac{i}{2} \sum_{i=1}^{n} \left(dz_i \wedge d\bar{z}_i + dw_i \wedge d\bar{w}_i \right), \tag{3.20}$$

and the vector field associated with a lie algebra element, e.g. an infinitesimal rotation is

$$X_u = i \sum_{i=1}^{n} \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - w_i \frac{\partial}{\partial w_i} + \bar{w}_i \frac{\partial}{\partial \bar{w}_i} \right) \tag{3.21}$$

computing the moment map via

$$d\langle \mu_I(z, wB), u \rangle = i_{X_u} \omega_I, \tag{3.22}$$

gives

$$i_{X_u} \omega_I = -\frac{1}{2} d \left(\|z\|^2 - \|w\|^2 \right) \tag{3.23}$$

so the moment map is

$$\mu_I(z, w) = -\frac{1}{2} \left(\|z\|^2 - \|w\|^2 \right). \tag{3.24}$$

We can compute the other Kähler forms using

$$\omega(U, V) = g(JU, V), \qquad \omega(U, V) = g(KU, V) \tag{3.25}$$

to give

$$\omega_J = \frac{i}{2}\sum_{i=1}^{n}\left(d\bar{w}_i \wedge d\bar{z}_i - dw_i \wedge dz_i\right), \tag{3.26}$$

$$\omega_K = \frac{1}{2}\sum_{i=1}^{n}\left(d\bar{w}_i \wedge d\bar{z}_i + dw_i \wedge dz_i\right) \tag{3.27}$$

Exercise to the reader check that these are correct. Hint: the conventions here are that $\omega(U,V) = i_V i_U \omega$.

Then we use $\omega_{\mathbb{C}} = \omega_J + i\omega_K = -i\sum_{i=1}^{n} d\bar{z}_i \wedge d\bar{w}_i$ to compute

$$\begin{aligned} i_{X_u}\omega_{\mathbb{C}} &= i\sum_{i=1}^{n}\left(i\bar{w}_i d\bar{z}_i + i\bar{z}_i d\bar{w}_i\right) \\ &= -d\sum_{i=1}^{n}\left(\bar{w}_i\bar{z}_i\right) = d\left(-\bar{w}\left(\bar{z}\right)\right). \end{aligned} \tag{3.28}$$

Thus the two moment maps are

$$\mu_I(z,w) = -\frac{1}{2}\left(\|z\|^2 - \|w\|^2\right), \qquad \mu_{\mathbb{C}}(z,w) = -\bar{w}\left(\bar{z}\right). \tag{3.29}$$

This does not quite agree with the details in [5] presumably due to some convention differences. I need to check this.

Now we can pick a regular point of μ, e.g. $\zeta = (1,0,0)$, and compute the quotient. Making use of the holomorphic symplectic version of the quotient this is

$$\mu_{\mathbb{C}}^{-1}(0,0)/U(1)_{\mathbb{C}} = \{(z,w) \in \mathbb{C}^n \oplus (\mathbb{C}^n)^* \ | z \neq 0, \ \ \bar{w}\left(\bar{z}\right) = 0\}/\mathbb{C}^*. \tag{3.30}$$

The $z \neq 0$ condition is because this is a fixed point of the $\mathbb{C}^*$ action so needs to be excluded to give the semistable points. How do we interpret what this space is?.

4 Week 4: 29/06/2022

Enric covered the "embryonic" and baby Nahm equations. These are respectively the construction of a bi-invariant on G and on its complexification $G_{\mathbb{C}} = T^*G$ which is the same as its cotangent bundle. This second relationship relies on the polar decomposition of elements of $G_{\mathbb{C}}$.

This material focuses on constructing the moduli space of solutions to the Nahm equations, and its metric, over the compact interval $I = [a,b]$ focussing particularly on $I = [0,1]$. Before discussing the most general case we start by treating some simpler examples. As well as [1] another nice reference for this material is [6].

Embryonic Nahm equations

As in previous weeks $\mathcal{A}$ is the space of connections on a G-bundle $P \to I = [0,1]$, and $\mathcal{G} \circlearrowleft \mathcal{A}$ is the action of the group of gauge transformations e.g. $g \in \mathcal{G}$ acts as

$$g \cdot A \in \mathcal{A} \mapsto gAg^{-1} - \dot{g}g^{-1}. \tag{4.1}$$

Here:

- $\mathcal{A} = C^1(I, \mathfrak{g})$ is a Banach space with the norm $\|A\| = \|A\|_\infty + \|\dot{A}\|_\infty$ where $\|A\|_\infty = \sup_{t\in I}|A(s)|$. In fact $\mathcal{A}$ is a Banach manifold with L^2 inner product

$$\langle X, Y\rangle = \int_I \langle X(t), Y(t)\rangle dt \tag{4.2}$$

where $X, Y \in T_A\mathcal{A} = \mathcal{A}$. This inner product is invariant under the $\mathcal{G}$ action.

- $\mathcal{G}$ is a Banach Lie group.

It is convenient to consider the gauge transformations which are the identity at the boundary[8]

$$\mathcal{G}^0 = \{g \in \mathcal{G} : g(0) = g(1) = 1\} \tag{4.3}$$

Not only does $\mathcal{G}^0$ act freely on $\mathcal{A}$ but it has finite dimensional slices, e.g. the space normal to the tangent space to the orbit is finite dimensional. There is a Kronheimer reference [7] which shows this explicitly for the case of the Nahm equations, for now we will take it on faith but I intend to include the full discussion once I have understood it. Thus $\mathcal{A}/\mathcal{G}^0$ is a finite dimensional manifold. There is a question about if this is a smooth manifold, this is claimed to be true but establishing this is not clear to me.

The residual action $\mathcal{G}/\mathcal{G}^0$ is identified with $G \times G$, since we are only left with the G's corresponding to the boundaries of the interval.

To establish all of this consider $A \in \mathcal{A}$ and write $g \in \mathcal{G}^0$ as $g = e^x$ for $x : I \to \mathfrak{g}$ with $x(0) = x(1) = 0$ e.g. $x \in \text{Lie}\left(\mathcal{G}^0\right)$ and acts on A by the adjoint action e.g.

$$g \cdot A = A + [x, A] - \dot{x} + \mathcal{O}(x^2). \tag{4.4}$$

The the tangent space to the orbit $\left(T_A\left(\mathcal{G}^0 \cdot \mathcal{A}\right)\right)$ is then has elements of the form $[x, A] - \dot{x}$. The slices of the $\mathcal{G}^0$ action are orthogonal to this. Thus $X \in T_A\mathcal{A}$ orthogonal to $T_A\left(\mathcal{G}^0 \cdot \mathcal{A}\right)$ if

$$\begin{aligned} 0 &= \int_I \langle X, [x, A] - \dot{x}\rangle dt \\ &= \int_I \left(\langle X, -\dot{x}\rangle + \langle [A, X], x\rangle\right) dt \\ &= \int_I \langle \dot{X} + [A, X], x\rangle dt, \end{aligned}$$

where we have integrated by parts going from the second to the third line with the boundary term vanishing due to $x(0) = x(1) = 0$. Thus the orthogonal complement to $T_A\left(\mathcal{G}^0 \cdot \mathcal{A}\right)$ is the space of solutions to the first order homogeneous linear ODE

$$\dot{X} + [A, X] = 0. \tag{4.5}$$

By standard theory the space of solutions is finite dimensional, and in fact is equal to $\dim\mathfrak{g}$.

An important issue to be aware of is that a single $\mathcal{G}^0$ orbit can intersect this orthogonal slice more than once, in physics this is known as the Gribbov ambiguity and causes all sorts of problems when trying to quantise gauge theories and identify their physical states. It can be shown[9] that if $b \in C^1(I, \mathfrak{g})$ is small enough then $\mathcal{G}^0 \cdot (A + b)$ intersects this slice of $T_A\mathcal{A}$ orthogonal to the gauge orbit only once.

[8] The claim is that this group acts freely on $\mathcal{A}$ but this is not obvious to me.
[9] Again we leave showing this until we can follow the details in [7].

The L^2 metric on $\mathcal{A}$ is invariant under $\mathcal{G}$ so descends to an L^2 metric on $\mathcal{M} = \mathcal{A}/\mathcal{G}^0$ that is $\mathcal{G}/\mathcal{G}^0 = G \times G$ invariant. We can turn this into a bi-invariant metric on G itself using the holonomy map.

The holonomy gives a map

$$\text{Hol} : \mathcal{A} \to G \tag{4.6}$$

via exponentiation. Being more explicit given $A \in \mathcal{A}$ we get a map $A : I \to \mathfrak{g}$ and there is a unique $g \in \mathcal{G}$ such that $g \cdot A = 0$ with $g(0) = 1$ e.g. the connection can be gauged away. This is equivalent to $\dot{g} = gA$ and solved by $g = \exp(tA)$ with $t \in I$. At $t = 1$ we get a unique element of G $\tilde{g} = g(1) = e^A$. This map is $\mathcal{G}^0$ invariant since $h \in \mathcal{G}^0$ satisfies $h(1) = 1$ so acting its action does not change $\tilde{g}$. Thus we have a map from $\mathcal{M} \to G$ which intertwines the respective $G \times G$ actions. This map turns out to be a diffeomorphism To see this we would need to consider the smooth structures more explicitly, at this stage it is definitely a bijection.

Baby Nahm equations

For the Baby Nahm equations we consider the complexification of the above story. e.g work with $\mathcal{A}_\mathbb{C} = \mathcal{A} \otimes_\mathbb{R} \mathbb{C}$, C^1 maps such that

$$A = A_0 + iA_1 : I = [0, 1] \to \mathfrak{g}_\mathbb{C}. \tag{4.7}$$

$\mathcal{A}_\mathbb{C}$ is an infinite dimensional Kähler manifold with the metric, complex structure, and Kähler form given by

$$\langle X, Y \rangle = \int_I (\langle X_0(t), Y_0(t) \rangle + \langle X_1(t), Y_1(t) \rangle)\, dt, \tag{4.8}$$

$$I : (X_0, X_1) \mapsto (-X_1, X_0), \tag{4.9}$$

$$\omega = \int_I dA_0 \wedge dA_1, \tag{4.10}$$

$$\omega(X, Y) = \langle IX, Y \rangle = \int_I (\langle -X_1, Y_0 \rangle + \langle X_0, Y_1 \rangle)\, dt, \tag{4.11}$$

for $X, Y \in T_A\mathcal{A}_\mathbb{C} = \mathcal{A}_\mathbb{C}$.

The action $\mathcal{G} \circlearrowleft \mathcal{A}_\mathbb{C}$ is given by

$$g \cdot (A_0, A_1) = \left(gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}\right), \tag{4.12}$$

Thus A_0 is a connection on our G-bundle P over I and A_1 is a section of $\operatorname{ad}(P)$ i.e a Higgs field. Again we will consider the normal subgroup $\mathcal{G}^0 \subset \mathcal{G}$ which acts freely on $\mathcal{A}_\mathbb{C}$. We now want to take the Kähler quotient of $\mathcal{A}_\mathbb{C}$ by $\mathcal{G}^0$, if these were both finite dimensional we could appeal to the results of the previous week . However, as they are infinite dimensional Banach manifolds we need to go through all the steps and check that works.

There is a moment map for the $\mathcal{G}^0 \circlearrowleft \mathcal{A}_\mathbb{C}$ action is

$$\mu : \mathcal{A}_\mathbb{C} \to C^0(I, \mathfrak{g}), \qquad A \mapsto \dot{A}_1 + [A_0, A_1]. \tag{4.13}$$

In fact the equation given by $\mu(A) = 0$ is what we are referring to as the baby Nahm equation, it follows from the full Nahm equations if $A_2 = A_3 = 0$. To explicitly construct the moment map consider that an element $x \in \text{Lie}\mathcal{G}^0$ defines a tangent vector

$$x^\#_A = \left(\frac{d}{ds} e^{sx} \cdot A\right)_{s=0} = ([x, A] - \dot{x}, [x, A_1]). \tag{4.14}$$

The moment map is then given by

$$i_{x_A^\#}\omega(Y) = d\langle \mu_A(Y), x\rangle, \tag{4.15}$$

where $Y \in T_A\mathcal{A}_\mathbb{C}$ Computing this we have

$$\begin{aligned} i_{x_A^\#}\omega(Y) &= \int_I (\langle -[x, A_1], Y_0\rangle + \langle [x, A_0] - \dot{x}, Y_1\rangle)\, dt \\ &= \int_I (\langle [Y_0, A_1] + [A_0, Y_1], x\rangle - \langle \dot{x}, Y_1\rangle)\, dt \\ &= \int_I \langle [Y_0, A_1] + [A_0, Y_1] + \dot{Y}_1, x\rangle dt, \end{aligned}$$

next note that

$$\begin{aligned} [Y_0, A_1] + [A_0, Y_1] + \dot{Y}_1 &= \lim_{\varepsilon\to 0} \Big([Y_0, A_1] + [A_0, Y_1] + \dot{Y}_1 + \varepsilon[Y_0, Y_1]\Big) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\dot{A}_1 + [A_0, A_1] + \varepsilon[Y_0, A_1] + \varepsilon[A_0, Y_1] + \varepsilon\dot{Y}_1 + \varepsilon^2[Y_0, Y_1] - \dot{A}_1 - [A_0, A_1]\right) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\dot{A}_1 + \varepsilon\dot{Y}_1 + [A_0 + \varepsilon Y_0, A_1 + \varepsilon Y_1] - \dot{A}_1 - [A_0, A_1]\right) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\mu\left(A + \varepsilon Y\right) - \mu(A)\right) \\ &= d\mu_A(Y) \end{aligned}$$

with $\mu(A) = \dot{A}_1 + [A_0, A_1]$ the desired moment map.

For a Kähler quotient we are interested in $\mu^{-1}(0)$, or a suitable level set, we can show that this is a Banach submanifold using the inverse function theorem if we can establish that μ is a submersion.

To do this consider $d\mu_A : T_A\mathcal{A}_\mathbb{C} = \mathcal{A}_\mathbb{C} \to C^0\left(I, \mathfrak{g}\right)$ which is surjective since $\forall x \in C^0(I, \mathfrak{g}), A \in \mathcal{A}_\mathbb{C}$ we can solve

$$\dot{X}_1 + [X_0, A_1] + [A_0, X_1] = x \tag{4.16}$$

for a unique $X \in T_A\mathcal{A}_\mathbb{C}$. We can always solve this equation since it is a first order linear inhomogeneous ODE. Thus $\mu^{-1}(0)$ is a Banach submanifold of $\mathcal{A}_\mathbb{C}$ Again since the $\mathcal{G}^0 \circlearrowleft \mathcal{A}_\mathbb{C}$ action is free $\mathcal{G}^0$ acts freely on the sub manifold $\mu^{-1}(0)$ and the argument from [7] that we are putting off tells us that the slice, e.g. the orthogonal complement to the $\mathbb{G}^0$ orbit, is finite dimensional. This means that $\mathcal{M} = \mu^{-1}(0)/\mathcal{G}^0$ is a finite dimensional Kähler manifold with a $G \times G = \mathcal{G}/\mathcal{G}^0$ action on it.

Once we know that $\mu^{-1}(0)/\mathcal{G}^0$ is a manifold showing that it is Kähler is fairly straight forward, the symplectic form on $\mathcal{A}_\mathbb{C}$ pulls back to $\mu^{-1}(0)$ via the inclusion map typically it will have degeneracies but these are coming from the tangent space to the orbit, as in the finite dimensional case. Carrying out the quotient removes these degenerate directions and we are left with a symplectic form on $\mathcal{M}$. The metric and complex structure likewise descend to $\mathcal{M}$ and can be shown to be compatible.

There is also an infinite dimensional analogue of the relationship between the Kähler quotient and the quotient of $\mathcal{A}_\mathbb{C}/\mathcal{G}^0_\mathbb{C}$ e.g. there is a biholomorphism

$$\mathcal{M} \to \mathcal{A}_\mathbb{C}/\mathcal{G}^0_\mathbb{C}. \tag{4.17}$$

Here $\mathcal{G}^0_\mathbb{C}$ means the normal subgroup of $\mathcal{G}_\mathbb{C} = C^2\left(I, G_\mathbb{C}\right)$ whose elements are the identity at the end points of I. This is proved by considering the $\mathcal{G}^0$-invariant Kähler potential on $\mathcal{A}_\mathbb{C}$,

$$f : \mathcal{A}_\mathbb{C} \to \mathbb{R}, \qquad f(A) = \frac{1}{2}\int_I \|A_1\| dt \tag{4.18}$$

which is compatible with the moment map μ.

As an exercise show that this is true e.g. show that $i_{Ix^{\#}_A} df = \langle \mu(A), x \rangle$.

A convexity argument applied to a Kempf-Ness type functional $F : \mathcal{G}^0_{\mathbb{C}} \to \mathbb{R}$ defined by $F(g) = f(g \cdot A)$ shows that $\forall A \in \mathcal{A}_{\mathbb{C}}\ \exists g \in \mathcal{G}^0_{\mathbb{C}}$ (unique up to a $\mathcal{G}^0$ action) such that $g \cdot A \in \mu^{-1}(0)$. This gives a homeomorphism between $\mathcal{M}$ and $\mathcal{A}_{\mathbb{C}}/\mathcal{G}^0_{\mathbb{C}}$. The holonomy map then gives a biholomorphism from $\mathcal{A}_{\mathbb{C}}/\mathcal{G}^0_{\mathbb{C}}$ to $G_{\mathbb{C}}$ which composes with the above homeomorphism to give a biholomorphism from $\mathcal{M}$ to $G_{\mathbb{C}}$ which intertwines the $G \times G$ actions. Thus we have a bi-invariant KäHler structure on the complexified group $G_{\mathbb{C}}$.

As a final step notice that the symplectic quotient $\mu^{-1}/\mathcal{G}^0$ can be identified with T^*G. To see this consider $g \in \mathcal{G}$ such that $g \cdot A_0 = 0$ with $g(0) = 1$. Since the moment map is $\mathcal{G}$ equivariant the connection $B = g \cdot A$ satisfies $\mu(B) = \dot{B}_1 = 0$ in other words $B = (0, x)$ for $x \in \mathfrak{g}$a constant. Then there is a diffeomorphism

$$\mu^{-1}/\mathcal{G}^0 \to G \times \mathfrak{g}, \qquad A \mapsto (g(1), A_1(0)), \tag{4.19}$$

and we can identify $G \times \mathfrak{g} = G \times \mathfrak{g}^*$using the invariant inner product and then use right translations to identify this with the cotangent bundle T^*G. This diffeomorphism is actually a symplectomorphism (This should be justified more carefully) which is compatible with the $G \times G$ action. We can go even further and see that the above Kähler potential descends to a KäHler potential on $\mathcal{M}$, and thus on T^*G as

$$\tilde{f} : G \times \mathfrak{g} \to \mathbb{R}, \qquad (g, x) \mapsto \frac{1}{2}\|\|^2. \tag{4.20}$$

Next time we will consider the quaternionised version of this story $\mathcal{A}_{\mathbb{H}} = \mathcal{A} \otimes \mathbb{H}$ with $A \in \mathcal{A}_{\mathbb{H}}$ given by

$$A = A_0 + iA_1 + jA_2 + kA_3 = A_0 + \Phi \tag{4.21}$$

where now the vanishing of the hyperkähler moment map for the $\mathcal{G}^0$ action gives the Nahm equations.

I have some more notes about this section so I may expand this section if I have time.

5 Week 5: 6/07/2022

A A sketch of the Nahm transform

I need to check that the conventions here match those in [1].

A.1 Monopole to Nahm

In case we do not cover anything about the Nahm transform I thought that it was worth giving a brief sketch of how the Nahm transform relates a monopole (Φ, A) to Nahm matrices $(A_0, \vec{A})$. This sketch follows the discussion in [8], [9] also has a nice discussion including explicitly carrying out the Nahm transform for a charge 1 monopole. Consider a Dirac spinor $\Psi(x)$ transforming in the fundamental representation of $SU(2)$ coupled to the charge N monopole (Φ, A). Write the spinor as

$$\Psi(x) = \begin{pmatrix} \Psi^- \\ \Psi^+ \end{pmatrix}, \tag{A.1}$$

and consider the Dirac equation

$$\begin{pmatrix} 0 & i\left(\vec{\tau} \cdot \vec{D} - i\Phi - s\right) \\ i\left(\vec{\tau} \cdot \vec{D} + i\Phi + s\right) & 0 \end{pmatrix} \begin{pmatrix} \Psi^- \\ \Psi^+ \end{pmatrix} = 0. \tag{A.2}$$

With $\vec{\tau}$ the vector of Pauli matrices, s a constant real parameter, and $D_A = d + \rho(A)\cdot$ the gauge covariant derivative in the representation ρ. This reduces to the two equations

$$D\Psi^- = 0 \qquad D^\dagger\Psi^+ = 0, \tag{A.3}$$

with

$$D = i\left(\vec{\tau}\cdot\vec{D} + i\Phi + s\right) \qquad D^\dagger = i\left(\vec{\tau}\cdot\vec{D} - i\Phi - s\right). \tag{A.4}$$

There are two vector spaces of L^2 normalisable solutions given by the kernels $\ker D$ and $\ker D^\dagger$. These vector spaces being non-empty depends on the asymptotics of $i\Phi + s$ which has eigenvalues $1+s, -1+s$. For solutions to decay in all directions, e.g. to be L^2, these need to have opposite signs[10] Thus $s \in [-1,1]$. Can show that $DD^\dagger \geq 0$ and thus D is surjective and $\ker D^\dagger = \emptyset$, then the Index theorem for $s \in [-1,1]$ gives

$$\mathrm{Ind}D = \dim\ker D = N. \tag{A.5}$$

The orthonormal basis for $\ker D$ is $\{\Psi^0 | 1 \leq a \leq N\}$ e.g.

$$\int \Psi_a^{0\dagger}(x)\Psi_b^0 d^3x = \delta_{ab}. \tag{A.6}$$

The three Nahm matrices are constructed from the zero modes as

$$(T_j)_{ab} = -i\int x^j\Psi_a^{0\dagger}(x)\Psi_b^0(x)d^3x \tag{A.7}$$

with $j = 1,2,3$. The zero modes depend on s and are chosen such that

$$\int x\Psi_a^{0\dagger}(x)\frac{\partial}{\partial s}\Psi_b^0(x)d^3x = 0, \tag{A.8}$$

this would be the connection piece A_0 which can be gauged to zero in $1D$.

A.2 Nahm to Monopole

The inverse Nahm transform works in essentially the same way. Gauge A_0 to be zero so that the Nahm data is a triple of $N\times N$ matrices $T_j(s)$ with $s \in [-1,1]$. There are two $1D$ Weyl equations

$$Dv^- = \left(\mathbb{I}_{2N}\frac{d}{ds} + iT_j\otimes\tau_j - \mathbb{I}_N\otimes x^j\tau_j\right)v^-(s) = 0, \tag{A.9}$$

$$D^\dagger v^+ = \left(-\mathbb{I}_{2N}\frac{d}{ds} + iT_j\otimes\tau_j - \mathbb{I}_N\otimes x^j\tau_j\right)v^+(s) = 0, \tag{A.10}$$

with $\vec{x}$ treated as a parameter. Since $DD^\dagger \geq 0$ the second equation has no solutions. Next we need to assume that the $T_j(s)$ have first order poles at $s = \pm 1$ with the matrix residues forming a N dimensional representation of $\mathfrak{su}(N)$. Thus $Dv^- = 0$ has two L^2 solutions with orthonormal basis of $\ker D$ $v_a(s)$ $a = 1,2$ such that

$$\int_{-1}^{1} v_a^\dagger v_b ds = \delta_{ab}. \tag{A.11}$$

Since the v_a vary smoothly with $\vec{x}$ we can use them to define Φ and A for a charge N monopole as

$$(\Phi(x))_{ab} = i\int_{-1}^{1} s v_a^\dagger v_b ds, \tag{A.12}$$

$$(A_j(x))_{ab} = \int_{-1}^{1} v_a^\dagger \frac{\partial}{\partial x^j} v_b ds. \tag{A.13}$$

[10] May need to give more details for why this is true.

Example A.1. Consider the case of $N = 1$ where the Nahm data is $T_j = ic_j$ for c_j a constant real vector. The $1D$ representation of $\mathfrak{su}(2)$ is the trivial representation so has no poles at $s = \pm 1$. Thus c_j corresponds to translations in $\mathbb{R}^3$. Consider the centred $N = 1$ monopole with $c_j = 0$, then

$$Dv(s) = \left(\frac{d}{ds} - \vec{\tau} \cdot \vec{x} \right) v(s) = 0. \tag{A.14}$$

This can be integrated to give

$$v(s) = \exp\left(s\vec{\tau} \cdot \vec{x} \right) v(0), \tag{A.15}$$

and the two independent orthonormal solutions are

$$v_1(s) = \sqrt{\frac{r}{\sinh 2r}} \left(\cosh sr + \sinh sr \hat{x} \cdot \vec{\tau} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.16}$$

$$v_2(s) = \sqrt{\frac{r}{\sinh 2r}} \left(\cosh sr + \sinh sr \hat{x} \cdot \vec{\tau} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.17}$$

The orthogonality condition is

$$\int_{-1}^{1} v_1^\dagger v_2 ds = 0. \tag{A.18}$$

Constructing the Higgs field gives

$$\begin{aligned} \Phi_{ab} &= i \frac{r}{\sinh 2r} \left(\hat{x} \cdot \vec{\tau} \right)_{ab} \int_{-1}^{1} 2s \cosh sr \sinh sr ds \\ &= i \left(\coth 2r - \frac{1}{2r} \right) \left(\hat{x} \cdot \vec{\tau} \right)_{ab} \end{aligned}$$

the Higgs field for the BPS monopole. The connection is recovered analogously.

References

[1] M. Mayrand. *Nahm's equations in hyperähler geometry*, Lecture notes, 2020.

[2] N. J. Hitchin. The Self-Duality Equations on a Riemann Surface. *Proc. London Math. Soc.*, (3) 55 (1987) 59-126.

[3] M. Atiyah and R. Bott. *The Yang-Mills Equations over Riemann Surfaces*, Phil. Trans. R. Soc. Lond. A. 308 (1983), 523–615.

[4] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles.* Princeton legacy library. Princeton Univercity Press, Princeton, July 2014.

[5] N. J. Hitchin. HyperKähler manifolds. Séminare Bourbaki, 206 (1992).

[6] R. Bielawski. Lie groups, Nahm's equationsand hyperkähler manifolds. Algebraic groups, pp. 1-17. Universitätsverlag Göttingen, Göttingen (2007). [arXiv:0509515]

[7] P. B. Kronheimer. A hyperkähler structure on the cotangent bundle of a complex Lie group. MSRI preprint (1998). [arXiv:0409253].

[8] N. S. Manton and P. M. Sutcliffe. *Topological solitons.* Cambridge monographs on mathematical physics. Cambridge University Press, Cambridge, July 2004.

[9] E. J. Weinberg and P. Yi, Magnetic Monopole Dynamics, Supersymmetry, and Duality, Phys. Rept. **438** (2007), 65-236 doi:10.1016/j.physrep.2006.11.002 [arXiv:hep-th/0609055 [hep-th]].