Notes on monopoles, mini twistors and spectral curves

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These notes are written to help me understand certain properties of monopoles. In particular I hope to sumarise the construction of mini-twistor space for monopoles due to Hitchin [1], spectral curves following [1, 2] and the rational map, particularly from the scattering point of view, following [1, 3]. Maybe I will include a discussion of the Moduli space following [4] if I have the time and inclination. A good general reference for monopoles is [5], however, it does not contain all the details for the topics that I want to include here.

1 Basics

We will be considering a three dimensional gauge theory with gauge group G, a matrix group. Our set up is a principal G bundle over a Riemannian 3-manifold M, $\pi : P \to M$. A monopole requires two things; a connection, A, on this principal bundle and a section, Φ, of an associated vector bundle,

$$\begin{array}{ccc} V & \longrightarrow & E \\ & & \downarrow \\ & & M \end{array} \tag{1.1}$$

where V is an n dimensional vector space and we have a representation of the Lie group $\rho : G \to GL(n, V)$. We will work with the case of ρ being the adjoint representation.

The action for the theory is

$$\mathcal{L} = \int_M \left(\mathrm{Tr}\left(\frac{1}{2} F \wedge \star F + \frac{1}{2} D\Phi \wedge \star D\Phi \right) + V(\Phi) d\mathrm{Vol} \right), \tag{1.2}$$

where

$$F = dA + \frac{1}{2}[A, A] \tag{1.3}$$

is the curvature of the connection A and

$$D\Phi = d\Phi + [A, \Phi], \tag{1.4}$$

is the covariant derivative of Φ. The third term,

$$V(\Phi) \tag{1.5}$$

is a potential term and we will usually be interested in minimisers of this potential. For Φ in the adjoint representation the equations of motion will be

$$D \star F = [\star D\Phi, \Phi], \tag{1.6}$$

$$D \star D\Phi = \frac{dV(\Phi)}{d\Phi}. \tag{1.7}$$

We also have to keep in mind that the curvature satisfies the Bianchi identity,

$$DF = 0. \tag{1.8}$$

We call pairs (A, Φ) which solve these equations G-monopoles. The most familiar examples being the $U(1)$ Dirac monopole and the $SU(2)$ t'Hooft-Polyakov monopole.

Typically the gauge group will be broken down to a sub group H which stabilises a vacuum expectation value of the Higgs Field, Φ_0, for which $V(\Phi_0) = 0$. To have an integrable energy density we need to impose the following asymptotics

$$\Phi|_{\partial M} \simeq \Phi_0, \quad (D\Phi)|_{\partial M} \simeq 0. \tag{1.9}$$

From the condition on the covariant derivative we can get a condition on the connection $A|_{\partial M}$.

1.1 BPS monopoles

We now specialise to solutions, (A, Φ), such that

$$V(\Phi) = 0, \tag{1.10}$$

As we still want an integrable energy density the same asymptotics still apply.

These are called BPS monopoles and correspond to limiting values of the couplings in the potentials, this is covered nicely in chapter 2 and 3 of [6]. In this case a completing the square argument can be used to find first order equations which imply the second order G-monopole equations. The argument proceeds as follows;

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\int_M \mathrm{Tr}\,(F \wedge \star F + D\Phi \wedge \star D\Phi), && (1.11)\\
&= \frac{1}{2}\int_M \mathrm{Tr}\,((F \pm \star D\Phi) \wedge \star(F \pm \star D\Phi) \mp 2F \wedge D\Phi), && (1.12)\\
&\geq \mp \int_M \mathrm{Tr}(F \wedge D\Phi), && (1.13)\\
&\geq \mp \int_{\partial M} \mathrm{Tr}(F\Phi)|_{\partial M}. && (1.14)
\end{aligned}$$

The first order equation

$$F = \mp \star D\Phi \tag{1.15}$$

is known as the Bogomolnyi equation and this bound is known as a Bogomolnyi bound. Consider that $\Phi_0 : \partial M \to G/H$, where G/H is the quotient of the gauge group by the stabiliser of Φ_0. We can show that

$$\mp \int_{\partial M} \mathrm{Tr}(F\Phi)|_{\partial M} \propto \deg(\Phi_0). \tag{1.16}$$

In fact as Φ_0 maps in to the coset space G/H which has distinct pieces the maps will fall into inequivalent classes and $\Phi_0 \in \pi_2(G/H)$.

For a semisimple gauge group, G, $\pi_2(G) = 0$ and we have that

$$\pi_2(G/H) \simeq \mathrm{Ker}\,(\pi_1(H) \to \pi_1(G))\,, \tag{1.17}$$

if further G is simply connected then $\pi_1(G) = 0$ and

$$\pi_2(G/H) = \pi_1(H). \tag{1.18}$$

Example 1.1. Specialise to the case of $M = \mathbb{R}^3$ and $G = SU(2)$ and $H = U(1)$, typically chosen to be the $U(1)$ generated by τ^3. In this case the boundary will be $\partial\mathbb{R}^3 \simeq S^2_\infty$, the two sphere at infinity and the asymptotics gives us a map

$$\Phi_0 : S^2\infty \to SU(2)/U(1) \simeq S^2, \tag{1.19}$$

so $\Phi_0 \in \pi_2(S^2) \simeq \mathbb{Z}$. The integral in the bound then becomes

$$\lim_{r\to\infty} \int_{S^2} \mathrm{Tr}(F\Phi)| = 4\pi \deg(\Phi_0). \tag{1.20}$$

This could also be interpreted as the flux of an asymptotic, abelian, magnetic field, $\mp\mathrm{Tr}(F|_{S^2_\infty}\Phi_0)$ through S^2_∞. We interpret this as the magnetic charge of the monopole. A specific example of a BPS monopole on $\mathbb{R}^3$ is the Prasad-Somerfield solution which asymptotes to the Dirac monopole.

2 Mini-twistor space

As geodesics in $\mathbb{R}^3$ are straight lines so I will just refer to line in this section, I will also use the term monopoles to mean BPS monopoles.

2.1 Constructing the twistor space

We follow [1, 2] here in taking the twistor space, T, to be the space of all oriented lines in $\mathbb{R}^3$. The correspondence space picture is the following

$$\begin{array}{ccc} & S^2 \times \mathbb{R}^3 & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} \\ T & & \mathbb{R}^3 \end{array} \tag{2.1}$$

The idea of the construction proceeds as follows for a point $\vec{x} \in \mathbb{R}^3$ the oriented lines through it are parametrised by an S^2 with centre $\vec{x}$. Each point $\vec{u} \in S^2$ gives the direction of the line. This results in the pair $(\vec{u}, \vec{x}) \in S^2 \times \mathbb{R}^3$. To find the twistor space we consider the flow along the straight line generated by a vector X which induces a map

$$\vec{x} \mapsto \vec{x} - (\vec{x}\cdot\vec{u})\vec{u} = \vec{v}, \tag{2.2}$$

This map gives a vector $\vec{v}$ orthogonal to $\vec{u}$ which is the closest approach of the line to the origin[1]

We thus have the following parameterisation for the straight line γ,

$$\gamma = \{\vec{y} \in \mathbb{R}^3 | \vec{y} = \vec{v} + t\vec{u} \quad \text{with } \vec{u}\cdot\vec{v} = 0, \quad t \in \mathbb{R}\}, \tag{2.3}$$

which gives us the twistor space

$$T = \{(\vec{u}, \vec{v}) \in S^2 \times \mathbb{R}^3 | \vec{u}\cdot\vec{v} = 0\}. \tag{2.4}$$

[1]To see this consider that a generic point on γ will be given by $\vec{x}+t\vec{u}$ for $t \in \mathbb{R}$ which has norm $|\vec{x}|^2+2t\vec{x}\cdot\vec{u}+t^2$. This is minimised when $t = -\vec{x}\cdot\vec{u}$.

This space is in fact nothing but the tangent bundle to the 2-sphere, TS^2. By identifying S^2 with $\mathbb{C}P^1$ in the ususal manner we have can extend the natural complex structure on TS^2, this is equivalent at a point (u, v) to taking the cross product with the normal direction u. To see this consider that a tangent vector to γ will be a pair $(\dot{u}, \dot{v})$ such that

$$u \cdot \dot{u} = 0, \qquad \dot{u} \cdot v + u \cdot \dot{v} = 0. \tag{2.5}$$

This leads to a field orthogonal to the geodesic $V = t\dot{u} + \dot{v} - (\dot{v} \cdot u)u$, which can be equivalently written as the pair of tangent vectors $(\dot{u}, \dot{v} - (\dot{v} \cdot u)u)$. It is on this pair of tangent vectors that taking the cross product with $\vec{u}$ is equivalent to the standard complex structure. In terms of complex coordinates a tangent vector will be of the form $w\frac{\partial}{\partial z}$ and we can give each line γ the complex coordinates (w, z).

A line, P_x through a fixed point $\vec{x} \in \mathbb{R}^3$ is defined by its direction $\vec{u}$ and is thus equivalent to a holomorphic section of $\pi : T \to \mathbb{C}P^1$. These sections are holomorphic vector fields on $\mathbb{C}P^1$ which can be expressed in terms of a quadratic polynomial[2]. The sections have the form

$$s(z) = \left(az^2 + bz + c\right) \frac{d}{dz}, \tag{2.6}$$

and are real, with respect to the real structure[3] $\tau(z) = \bar{z}^{-1}$, if and only if

$$a = -\bar{c}, \qquad b = \bar{b}. \tag{2.7}$$

Checking this is left as an exercise to the reader but is discussed in [1].

This means that the point $\vec{x} \in \mathbb{R}^3$ is represented by the real section

$$s(z) = \left(\left(x^1 + ix^2\right) - 2x^3 z - \left(x^1 - ix^2\right) z^2\right) \frac{d}{dz}. \tag{2.8}$$

In [1] it is noted that the space T should really be interepreted as an affine bundle over $\mathbb{C}P^1$, then a choice of a zero section is equivalent to choosing an origin in $\mathbb{R}^3$.

2.2 Constructing monopoles from twistor space

To construct a $SU(2)$ monopole on $\mathbb{R}^3$ we need to consider a principal $SU(2)$ bundle, P, over $\mathbb{R}^3$ with connection $D_A = d + A$, and a section of the adjoint bundle Φ. These give a BPS monopole if the curvature of A is related to the Higgs field Φ through

$$F = \pm \star D_A \Phi. \tag{2.9}$$

If we take E to be the associated rank 2 vector bundle to P then following [1] define a rank 2 vector bundle $\tilde{E}$ on the twistor space T as

$$\tilde{E}_p = \{s \in \Gamma(\gamma_p, E) | \left(i_U D_A - i\Phi\right) s = 0\}. \tag{2.10}$$

In the above definition γ_p is the oriented geodesic corresponding to a point $p \in T$ and U is the unit tangent vector to γ_p. As we have an ODE along each line, the finite dimensional space of solutions gives the fibre of $\tilde{E}$ over each point $p \in T$.

With this definition in hand in Hitchin proves the following theorem

[2] This is because the tangent bundle has degree 2

[3] This real structure is minus the antipodal map.

Theorem 2.1 (Theorem (4.2) in [1]). *If (A, Φ) solve the $SU(2)$ BPS equations, then $\tilde{E}$ is in a natural way a holomorphic vector bundle over T such that*

1. *$\tilde{E}$ is trivial on every real section.*
2. *$\tilde{E}$ has a symplectic structure.*
3. *$\tilde{E}$ has a quaternionic structure,*
$$\sigma : \tilde{E}_p \to \tilde{E}_{\tau(p)}, \tag{2.11}$$
with $\sigma^2 = -1$.

Conversely, every such holomorphic vector bundle on T defines a BPS monopole.

The proof of this theorem is given in [1] and will not be replicated here. In fact the theorem can be extended to hold for any of the standard Lie groups[4].

Example 2.2 ($U(1)$ monopoles). Here the bundle E over $\mathbb{R}^3$ is taken be a line bundle with a flat connection, e.g. A is pure gauge, and constant Higgs field $\Phi = i$. The theorem tells us that we should construct a holomorphic line bundle over the twistor space $L \to T$.

We define L fibre wise above a point $p \in T$ as

$$L_p = \{s \in C^\infty(\gamma_p) | \frac{ds}{dt} + s = 0\}. \tag{2.12}$$

Note that since E is a line bundle we have that $\Gamma(\gamma_p, E) \simeq C^\infty(\gamma_p)$ so the definition of L_p is just a special case of the definition of $\tilde{E}_p$ above. The solutions of the ODE are of the form $s = Ae^{-t}$, for A a constant.

On the correspondence space $S^2 \times \mathbb{R}^3$ define the function

$$\hat{l} : (\vec{u}, \vec{x}) \mapsto e^{-\vec{u}\cdot\vec{x}}, \tag{2.13}$$

on a straight line[5], $\vec{x} = \vec{y} + t\vec{u}$, $\hat{l}(\vec{u}, \vec{x}) = e^{-\vec{y}\cdot\vec{u}}e^{-t}$ so $\hat{l}$ defines a global non-vanishing section of the line bundle $L \to T$. It is important to bear in mind that l depends on the choice of origin.

Now that we have global sections of L we want to compute $\bar{\partial} l$. The first thing to note is that if $\vec{u} \cdot \vec{x} = 0$ then $\hat{l} = 1$ so we can see that $\bar{\partial} l = 0$ on the horizontal piece of $S^2 \times \mathbb{R}^3$, that is the piece orthogonal to the S^2 fibres. However, this horizontal piece becomes the fibre of $\pi : T \to S^2$ which means that $\bar{\partial} l = 0$ in the fibre direction.

To check that it also vanishes in the S^2 fibre direction of the correspondence space we use the standard complex coordinates $z = x + iy$. Using stereographic projection from the south pole of the two-sphere we find that

$$\vec{u} = \begin{pmatrix} \frac{2x}{1+|z|^2} \\ \frac{2y}{1+|z|^2} \\ \frac{1-|z|^2}{1+|z|^2} \end{pmatrix}. \tag{2.14}$$

[4] The reality condition apparently needs to be modified and the group is required to be a real form of a complex Lie group.

[5] Note that in the parameterisation of geodesics we were using above $\vec{x} = \vec{v} + s\vec{u}$ where the first piece $\vec{v}$ vanishes when dotted with $\vec{u}$ however, the parameter s may not be the same as t but will differ by addition of a constant.

Expressing the vector $\vec{x}$ as

$$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{2.15}$$

and writing $a = x^1 + ix^2$, $b = x^3$ we arrive at

$$\vec{u} \cdot \vec{x} = \frac{a\bar{z} + \bar{a}z + b\left(1 - |z|^2\right)}{1 + |z|^2}. \tag{2.16}$$

In the proof of the theorem Hitchin defines the $\bar{\partial}$ operator through the connection $\nabla^{0,1}$ on correspondence space which acting on $\vec{u} \cdot \vec{x}$ will only have a $\frac{\partial}{\partial \bar{z}} \wedge d\bar{z}$ piece[6]. Computing this we have

$$\nabla^{0,1}\left(\vec{u} \cdot \vec{x}\right) = \left(a - 2bz - \bar{a}z^2\right) \frac{d\bar{z}}{\left(1 + |z|^2\right)^2}. \tag{2.17}$$

In the complex coordinates $(w, z) \mapsto w\frac{d}{dz}$ for T interpreted as TS^2 comparison with the real section corresponding to the point $\vec{x} \in \mathbb{R}^3$, Equation (2.8), we have that

$$w = a - 2bz - \bar{a}z^2. \tag{2.18}$$

This gives the equation for the projective line P_x which corresponds to the point $\vec{x}$[7]. From this we have that the section l satisfies[8]

$$\bar{\partial}l = \left(\nabla^{0,1}\hat{l}\right)' = -\frac{wd\bar{z}}{\left(1 + |z|^2\right)^2} l. \tag{2.19}$$

To understand more about L consider a local holomorphic section fl, as $\bar{\partial}(fl) = 0$ we have that

$$\frac{\partial f}{\partial \bar{w}} = 0, \qquad \frac{\partial f}{\partial \bar{z}} = \frac{wf}{\left(1 + |z|^2\right)^2}, \tag{2.20}$$

which leads to

$$f = g(w, z)e^{-\frac{w}{z(1+|z|^2)}}, \tag{2.21}$$

for g holomorphic. In local patches $U_\pm$ where

$$U_- = \{(w, z) \in T | z \neq 0\}, \quad U_+ = \{(w, z) \in T | z \neq \infty\}, \tag{2.22}$$

we have the functions

$$f_- = e^{-\frac{w}{z(1+|z|^2)}}, \qquad \text{regular at } z = \infty \text{ but singular at } z = 0, \tag{2.23}$$

and

$$f_+ = e^{-\frac{w}{z(1+|z|^2)} + \frac{w}{z}}, \qquad \text{regular at } z = 0 \text{ but singular at } z = \infty. \tag{2.24}$$

[6]This is because $\bar{\partial}l = 0$ along the S^2 fibre.

[7]Note that the section $w = 0$ corresponds to the origin of $\mathbb{R}^3$.

[8]We are using a prime here to denote transforming a function on the correspondence space in to a section of L. This is how the $\bar{\partial}$ operator is defined in [1].

We thus have $f_\pm l$ as a trivialisation of L on $U_\pm$. On the intersection we have that

$$f_- l = e^{-\frac{w}{z}} f_+ l, \tag{2.25}$$

so that the transition functions are

$$\phi_{+-}(w,z) = e^{-\frac{w}{z}}. \tag{2.26}$$

In these coordinates the real structure, τ, on T is

$$\tau(w,z) = \left(-\frac{\bar{w}}{\bar{z}^2}, -\frac{1}{\bar{z}}\right), \tag{2.27}$$

this interchanges $U_\pm$ and it can be seen that

$$\phi_{+-} \circ \tau(w,z) = e^{\frac{\bar{w}}{\bar{z}}} = \bar{\phi}_{+-}^{-1}(w,z), \tag{2.28}$$

which gives an antiholomorphic isomorphism $\sigma : L_p \to L^*_{\tau p}$.

The line bundle L is sometimes called the exponential line bundle.

So far we have not imposed any boundary conditions on our solutions to the BPS equations, if we do we can define the spectral curve of a monopole.

3 Spectral curves

From a physical viewpoint the boundary conditions we shall impose on a monopole are the requirement that the energy is finite, this corresponds to having the fields Φ, A tend to a vacuum value as $|\vec{x}| \to \infty$. This corresponds to picking a direction in the Lie algebra, Φ^∞, which breaks the symmetry from G to the stabaliser of Φ^∞. Here we will follow [2] and only consider the case of maximal symmetry breaking where G breaks to its maximal torus T. For Φ the asymptotics are taken to be

$$\Phi = \Phi^\infty + \psi\frac{1}{r} + O\left(\frac{1}{r^2}\right), \tag{3.1}$$

where Φ^∞ is valued in G/T and $\psi : S^2_\infty \to K/T$. In [2] some examples of maximal tori for particular Lie groups are given.

Working in spherical polar coordinates and a radially symmetric gauge, $A_r = x^i A_i = 0$, the BPS equations give that

$$\partial_r \Phi = \frac{1}{r^2 \sin\theta} F_{\theta\varphi}. \tag{3.2}$$

Differentiating the boundary conditions, Equation (3.1), and imposing

$$\frac{1}{\sin\theta} F_{\theta\varphi} = \star F^\infty_{\theta\varphi} + O\left(\frac{1}{r}\right), \tag{3.3}$$

we find that[9]

$$\Phi = \Phi^\infty - \star F^\infty_{\theta\varphi}\left(\frac{1}{r}\right) + O\left(\frac{1}{r^2}\right). \tag{3.4}$$

[9] Here we are just comparing the coefficients at a given order of $\frac{1}{r}$ and find that at order $r^0 : \partial_r \Phi^\infty = 0$, at order $r^{-1} : \partial_r \psi = 0$ and at order $r^{-2} : \star F^\infty_{\theta\varphi} = -\psi$.

The case that will be of particular interest to us will be when $G = SU(n)$ where the maximal torus T is the traceless diagonal $n \times n$ matrices. Here the asymptotics become

$$\Phi = i\mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right) - \frac{i}{2r}\mathrm{diag}\left(k_1, \ldots, k_n\right) + O\left(\frac{1}{r^2}\right), \tag{3.5}$$

here the traceless condition means that $\sum \lambda_i = \sum k_i = 0$ and we also assume that the eigenvalues of Φ^∞ are ordered such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and $k_i \in \mathbb{Z}$[10].

Before discussing the spectral curve for general n we will first consider the case $n = 2$.

3.1 Gauge group $SU(2)$

The key to the construction of the spectral curve is the equation

$$\left(i_U D_A - i\Phi(\vec{x})\right) v(\vec{x} = 0), \tag{3.6}$$

for $\vec{x} \in \gamma$ and $v(\vec{x} \in \mathbb{C}^2$, which we encountered before when going between bundles over $\mathbb{R}^3$ and T. If there is a parameter t along the line we can write this as

$$\left(\frac{d}{dt} + u^i A_i(t) - i\Phi(t)\right) v(t) = 0. \tag{3.7}$$

If we plug in the boundary conditions, Equation (3.5) for $n = 2$[11], and keep working in the radial gauge where $u^i A_i = 0$ we arrive at

$$\frac{dv}{dt} + \begin{pmatrix} \lambda - \frac{k}{2t} & 0 \\ 0 & -\lambda + \frac{k}{2t} \end{pmatrix} v + O\left(\frac{1}{t^2}\right) v = 0. \tag{3.8}$$

This equation will have two solutions, $v_1(t), v_2(t)$ such that

$$v_1(t) t^{-\frac{k}{2}} e^{\lambda t} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2(t) t^{\frac{k}{2}} e^{-\lambda t} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{as } t \to \infty. \tag{3.9}$$

This means that there is a one-dimensional subspace $L^+(\gamma) \subset \tilde{E}$ of solutions decaying exponentially[12] as $t \to \infty$. Changing variables to $s = -t$ we see that there is another one-dimensional subspace $L^-(\gamma) \subset \tilde{E}$ of solutions decaying at $s \to \infty$, negative infinity with respect to t.

Definition 3.1. *The spectral curve S is the set of lines γ such that*

$$L^+(\gamma) = L^-(\gamma). \tag{3.10}$$

It is noted in [1] that the antiholomorphic structure $\sigma : \tilde{E} \to \tilde{E}$ sends $L^+ \to L^-$ and gives $L^- = L^+(-k)$. This is used to project $L^- \subset \tilde{E}$ onto $(L^+)^*$ and obtain a holomorphic section

$$\psi \in H^0\left(T, \left(L^+ \otimes L^-\right)^*\right) \simeq H^0\left(T, O(2k)\right). \tag{3.11}$$

The zero set of ψ corresponds to $L^+ = L^-$, the spectral curve S.

In [1] the following proposition is proved:

[10] This is because they are the Chern number of the piece of F^∞ along each of the n asymptotic $U(1)$ bundles.

[11] We take $\lambda_1 = -\lambda_2 = \lambda$ and $k_1 = -k_2 = k$.

[12] N.b. $v_1(t)$ decays exponentially as it needs to be multiplied by $e^{\lambda t}$ for $\lambda > 0$ to get a constant vector asymptotically.

Proposition 3.2 (Proposition (7.3) in [1])**.** *The spectral curve S of a charge k $SU(2)$ monpopole has the following properties:*

1. *S is compact.*
2. *S is defined by the equation*
$$p(w,z) = w^k + a_1(z)w^{k-1} + \cdots + a_k(z) = 0, \tag{3.12}$$
with $a_i(z)$ a degree $2i$ polynomial in z.
3. *The line bundle L^2 is holomorphically trivial on S.*
4. *S is preserved by the real structure on τ.*

I will sketch why the spectral curve is defined by the zero set of a polynomial but the rest of the details are given in [1]. The idea is that for a polynomial $p(w,z)$ as in the theorem

$$\psi = p(w,z)\left(\frac{d}{dz}\right)^k \tag{3.13}$$

will be a holomorphic section of $O(2k)$, and that the dimension of the space of such sections is $(k+1)^2$. Next it can be checked that $\dim H^0(T, O(2k)) \leq (k+1)^2$, which means that all sections must be off this form. Compactness of S then implies that the coefficient of w^k is non-zero[13].

We now consider an example.

Example 3.3 (Charge one monopole)**.** Take $k = 1$ then the polynomial which defines the spectral curve will be

$$w + a_1(z) = 0 \tag{3.14}$$

for $a_1(z)$ of degree two. However, this is just a section of $\pi : T \to S^2$ and as S is real corresponds to some projective line P_x and thus a specific point $\vec{x} \in \mathbb{R}^3$. In fact in [1] it is shown that if we pick the origin of $\mathbb{R}^3$ we get the BPS monopole, and solving the scattering equation for lines through the origin $\frac{dx}{dr} - \Phi(r)x = 0$ we find the unique solution

$$x = \begin{pmatrix} 0 \\ \frac{r}{\sinh(r)} \end{pmatrix}, \tag{3.15}$$

which decays as $r \to \pm\infty$ which gives us that $S = P_x$. This also tells us that the BPS monopole is the unique charge 1 solution to the BPS equations.

Is it worth including the axially symmetric example from Hitchin as well? Also should I spell out more of the details from the first example?

[13] I do not find this clear so need to think about it.

3.2 Gauge group $SU(n)$

Now following [2] we can start to consider how to define the spectral curve for a higher rank unitary group $SU(n)$. In principal we could talk about any of the classical groups but it is easiest to stick to unitary groups for now[14] The idea is to consider a group $G = SU(n)$ and an irreducible representation V of G. Usually V will be taken to be the fundamental representation but it doesn't have to be. The key idea is that each representation will have a highest weight state, denote the representation by V_λ. Then we can construct the space $E_\lambda(\gamma)$ of solutions to

$$(D_t - i\Phi)\, v(t) = 0, \tag{3.16}$$

along the line γ with $v(t)$ valued in V_λ. If there is an inner product $\langle , \rangle$ on V_λ then considering $w(t) \in E_\lambda(-\gamma), v(t) \in E_\lambda(\gamma)$ we have, using Equation (3.16) along γ and $(D_t + i\Phi)\, w(t) = 0$ along $-\gamma$, that

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \langle D_t v, w\rangle + \langle v, D_t w\rangle, \tag{3.17}$$

$$= \langle i\Phi v, w\rangle + \langle v, -i\Phi w\rangle, \tag{3.18}$$

$$= \langle v, -i\Phi^* w - i\Phi w\rangle, \tag{3.19}$$

$$= 0, \tag{3.20}$$

using $\Phi^* + \Phi = 0$ in $\mathfrak{su}(V_\lambda)$. This give a pairing

$$\langle , \rangle : E_\lambda(\gamma) \times E_\lambda(-\gamma) \to \mathbb{C}. \tag{3.21}$$

Each of the weights, μ in V_λ will correspond to an eigenvector[15] $e(\mu)$ such that there is a $v(t) \in E_\lambda(\gamma)$ satisfying

$$v(t)t^{-\frac{\mu(\star F^\infty)}{2}} e^{\mu(\Phi^\infty)t} \to e(\mu) \quad \text{as } t \to \infty. \tag{3.22}$$

By considering the highest weight λ we can define the one-dimensional subspace

$$E^+_\lambda(\gamma) = \{v \in E_\lambda(\gamma) | \|v(t)t^{-\frac{\lambda(\star F^\infty)}{2}} e^{\lambda(\Phi^\infty)t}\| \text{ is bounded as } t \to \infty\}. \tag{3.23}$$

The other weights will can be used to define the corresponding $\dim V_\lambda - 1$ dimensional subspace

$$E^-_\lambda(\gamma) = \{v \in E_\lambda(\gamma) | \|v(t)t^{-\frac{\mu(\star F^\infty)}{2}} e^{\mu(\Phi^\infty)t}\| \text{ is bounded as } t \to -\infty, \forall \mu < \lambda\}, \tag{3.24}$$

where we have used the ordering on the weights to say that $\mu < \lambda$. The decay rates of the elements of these subspaces then give that

$$\langle E^+_\lambda(\gamma), E^-_\lambda(-\gamma)\rangle = 0. \tag{3.25}$$

We can now define the spectral curve associated to the representation with highest weight λ.

[14] I am also much more comfortable with the sections of [2] that just focus on unitary groups.

[15] I think that these are eigenvector of the Cartan of $\mathfrak{su}(V_\lambda)$. We need the same ODE theory as in the $SU(2)$ case but now we have that V_λ splits into weight spaces and each weight space gives a solution to Equation (3.16).

Definition 3.4. *The spectral curve S_λ is the real algebraic curve given by the set of lines, γ, such that*

$$E_\lambda^+(\gamma) \subset E_\lambda^-(\gamma). \tag{3.26}$$

If the weight λ can be decomposed as a sum of fundamental weights $\lambda = \sum_i n_i \mu_i$, $n_i \in \mathbb{Z}_+$, then the spectral curve can be decomposed as the sum

$$S_\lambda = \sum_i n_i S_i. \tag{3.27}$$

We will call $S_i = S_{\mu_i}$ the i'th spectral curve of the monopole. We can define topological weights for the monopole using $\star F^\infty$ as

$$m_i = \mu_i \left(\star F^\infty\right). \tag{3.28}$$

In [2] The degree of S_i is shown to be $2m_i$ and thus S_λ has degree $4\lambda(\star F^\infty) = \sum_i n_i m_i$.

We can consider embedding a $k = 1$, $SU(2)$ monopole along a simple root, this is called a fundamental monopole and means that m_i will only be non-zero for the fundamental weight which corresponds to this simple root, α_i. From Example 3.3 above we know that S_i corresponds to the lines through the monopoles centre, the projective line P_x, and as the other topological weights are zero their spectral curves will be empty.

This next part is pretty much a direct lift from [2] but I will try to add some comments. Working in the fundamental representation of $SU(n)$ we can say a little more. Here we know that the asymptotics of the Higgs field, Equation (3.5), along a line γ are

$$\Phi = i\mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right) - \frac{i}{2r}\mathrm{diag}\left(k_1, \ldots, k_n\right) + O\left(\frac{1}{r^2}\right), \tag{3.29}$$

where the matrices are traceless and the λ_i are ordered as above. Then there will be a plus subspace defined for each λ_i as

$$E_i^+(\gamma) = \{v \in E(\gamma) | \|v(t) t^{-\frac{k_i}{2}} e^{\lambda_i t}\| \text{ is bounded as } t \to +\infty\}, \tag{3.30}$$

and the minus subspace as

$$E_{n-i+1}^-(\gamma) = \{v \in E(\gamma) | \|v(t) t^{-\frac{k_i}{2}} e^{\lambda_i t}\| \text{ is bounded as } t \to -\infty\}. \tag{3.31}$$

This time the pairing will satisfy

$$\langle E_i^+(\gamma), E_{n-i}^-(-\gamma)\rangle = 0, \qquad \langle E_i^-(\gamma), E_{n-i}^+(-\gamma)\rangle = 0, \tag{3.32}$$

and we define

$$T_i = \{\gamma | E_i^+(\gamma) \cap E_{n-i}^-(\gamma) \neq 0\}. \tag{3.33}$$

An alternative picture of T_i is as the collection of γ such that the image

$$\Lambda^i E_i^+(\gamma) \to \Lambda^i \left(E / E_{n-i}^-(\gamma)\right) \tag{3.34}$$

vanishes. In fact $S_i = T_i$ gives the i'th spectral curves. A constraint is needed on $S_i \cap S_j$, that it is finite whenever the i'th and j'th simple roots are joined on the Dynkin diagram. The decomposition of $E(\gamma)$ in terms of the $E_i^+(\gamma)$ gives a flag decomposition

$$E_1^+(\gamma) \subset E_2^+(\gamma) \subset \cdots \subset E_n^+(\gamma) = E(\gamma) \tag{3.35}$$

from which E can be built up using the i'th spectral curve at each step.

3.2.1 Examples

We are now able to see some examples of how to construct the spectral curve for $SU(2)$ and $SU(3)$ monopoles when V is taken to be the fundamental representation. Completely explicit constructions will not be given as the form of A would be needed for that. According to [2] for the exceptional groups it is better to work with the adjoint representation which adds complications.

Example 3.5. We have already encountered the standard definition of the spectral curve when the Gauge group is $SU(2)$ but now we want to see that the more general definition for $SU(n)$ reduces to the familiar case when V is the fundamental representation of $SU(2)$. Here the asymptotic expression for the Higgs field is

$$\Phi = i\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix} - \frac{i}{2r}\begin{pmatrix}k & 0\\ 0 & -k\end{pmatrix} + O\left(\frac{1}{r^2}\right). \tag{3.36}$$

There will be two weights, $\lambda_\pm$ such that $\lambda_\pm(\star F^\infty) = \pm k, \lambda_\pm(\Phi^\infty) = \pm\lambda$. Following standard results on ODEs we can see that for an eigenvector $e(\lambda_\pm) \in V(\lambda_\pm)$ that there will be a function $v(t) \in E(\gamma)$ such that

$$v(t)t^{-\frac{\lambda_\pm(\star F^\infty)}{2}}e^{\lambda_\pm(\Phi^\infty)t} \to e(\lambda_\pm) \quad \text{as } t \to \infty. \tag{3.37}$$

In this case λ_+ is the highest weight and the line bundle of interest is

$$\begin{aligned}
E^+_{\lambda_+}(\gamma) &= \{v \in E(\gamma)|\ \|v(t)t^{-\frac{1}{2}\lambda_+(\star F^\infty)}e^{\lambda_+(\Phi^\infty)t}\| \text{ is bounded as } t \to \infty\},\\
&= \{v \in E(\gamma)|\ \|v(t)t^{-\frac{k}{2}}e^{\lambda t}\| \text{ is bounded as } t \to \infty\},\\
&= L^+(\gamma).
\end{aligned}$$

By writting out the definitions in this way we also see that in this case

$$E^-_{\lambda_+} = L^-(\gamma). \tag{3.38}$$

As these are both 1 dimensional subspaces the condition that the spectral curve is given by

$$S = \{\gamma | E^+_{\lambda_+}(\gamma) \subset E^-_{\lambda_+}(\gamma)\} \tag{3.39}$$

reduces to the familiar condition

$$S = \{\gamma | L^+(\gamma) = L^-(\gamma)\} \tag{3.40}$$

Example 3.6. If we take V to be the fundamental $\underline{3}$ representation of $su(3)$, which has highest weight

$$\lambda = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right), \tag{3.41}$$

then $\dim E^-_\lambda = 2$. The weight space for this representation of $SU(3)$ is given in Figure 1. We could also take the anti-fundamental $\overline{\underline{3}}$ and find the same thing.

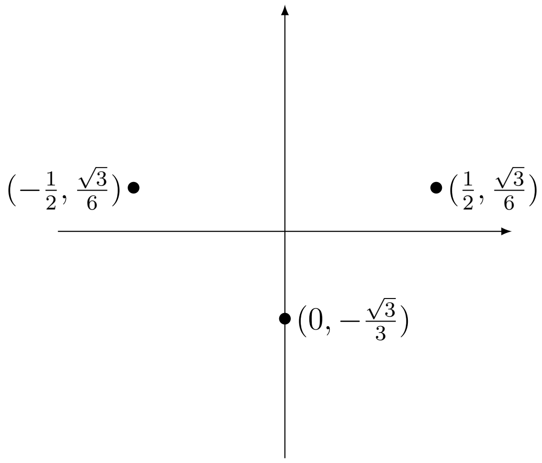

Figure 1: The weight lattice for the $\underline{3}$ of $SU(3)$ with highest weight $(\frac{1}{2}, \frac{\sqrt{3}}{6})$, the $\underline{\bar{3}}$ is the reflection of this in the y-axis.

TO make contact with the more explicit construction of the spectral curve for $SU(n)$ in its fundamental representation recall that the Higgs field has the following form

$$\Phi = i \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix} - \frac{i}{2r} \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & -k_1 - k_2 \end{pmatrix} + O\left(\frac{1}{r^2}\right). \tag{3.42}$$

The subspaces used to construct S in this case are

$$E_1^+(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{-\frac{k_1}{2}} e^{\lambda_1 t}\| < \infty, t \to \infty\}, \tag{3.43}$$

$$E_2^+(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{-\frac{k_2}{2}} e^{\lambda_2 t}\| < \infty, t \to \infty\}, \tag{3.44}$$

$$E_3^+(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{\frac{k_1+k_2}{2}} e^{-\lambda_1 t - \lambda_2 t}\| < \infty, t \to \infty\}, \tag{3.45}$$

$$E_1^-(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{\frac{k_1+k_2}{2}} e^{-\lambda_1 t - \lambda_2 t}\| < \infty, t \to -\infty\}, \tag{3.46}$$

$$E_2^-(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{-\frac{k_2}{2}} e^{\lambda_2 t}\| < \infty, t \to -\infty\}, \tag{3.47}$$

$$E_3^-(\gamma) = \{v(t) \in E(\gamma) | \|v(t)t^{-\frac{k_1}{2}} e^{\lambda_1 t}\| < \infty, t \to -\infty\}. \tag{3.48}$$

Due to the ordering on the λ_i we immediately see that $E_1^+ \subset E_2^+ \subset E_3^+$, you should also convince yourselves that $E_3^+ = E$, to see this go back to the definition of $E(\gamma)$ in terms of solutions to the scattering equation. Also $E_1^- \subset E_2^- \subset E_3^- = E$ for the same reason.

There will be two spectral curves S_1 when $E_1^+ \subset E_2^-$ and S_2 when $E_1^- \subset E_2^+$.

4 Monopoles and rational maps

A nice observation is made about the rational map of an $SU(2)$ monopole in [4]. We saw that that there are two solutions, $v_1(t), v_2(t)$ to the scattering equation, Equation (3.16), along a straight line γ with $v_1(t)$ decaying exponentially as $t \to \infty$. However, there will also be a solution $v_1'(t)$ which decays exponentially as $t \to -\infty$. This can be written as a linear combination of the v's

$$v_1'(t) = pv_1(t) + qv_2(t). \tag{4.1}$$

To each line γ we then assign

$$R(\gamma) = \frac{p}{q} \in \mathbb{C}P^1 \tag{4.2}$$

Called the scattering along γ. It is a theorem due to Donaldson which states that for a given charge k, $SU(2)$, monopole the scattering is a based rational function of degree k. In fact it is in [3] where the interpretation of the Donaldson rational map in terms of the scattering data is given. To understand this a little better split $\mathbb{R}^3$ into $\mathbb{C} \times \mathbb{R}$ where we take $\zeta = x^1 + ix^2$ to be our complex coordinate. If we consider lines through the ζ-plane parallel to the x^3-axis then we can take $t = x^3$ in the scattering equation and we will find that p, q are functions of ζ. This results in

$$R(\zeta) = \frac{p(\zeta)}{q(\zeta)} \tag{4.3}$$

being the Donaldson rational map.

5 Moduli space

This section needs to be written! I can use [5] as a reference for the $SU(2)$ case and the geodesic approximation.

1. Metric on the moduli space and the geodesic approximation.
2. Hyperkähler construction in terms of moment maps.
3. Euclidean $SU(2)$ monopoles, explicit construction of the charge two moduli space metric and Sen form.

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