Moment maps and Hyperkähler quotients

Notes C Ross [1], March 2017

These are some notes on moment maps and hyperkähler quotients that I wrote to help me in preparing for a talk at the BIG workshop on Higgs Bundles[2]. The end goal is to have a construction where the self-duality equations can be viewed as moment maps and the Moduli space of solutions to the self-duality equations can then be constructed as a hyperkähler quotient. The main referenece is [1] though [2] was particularly useful for an explicit example of a moment map computation, the "real" case considered in Example 1.2 below.

1 Moment maps

Moment maps appear whenever we have a Lie group, G acting on a symplectic space, M, where the group action leaves the symplectic form ω fixed. We can construct a map

$$\mu : M \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, through

$$\langle \mu(x), \zeta \rangle = f_{X_\zeta}(x)$$

where $\zeta \in \mathfrak{g}$, $x \in M$, X_ζ is the vector field generated by ζ, f_{X_ζ} is the Hamiltonian function for the vector field X_ζ and $<,>$ is the natural pairing between a Lie algebra and its dual. Then using the symplectic form we can turn this into the one form

$$d\langle \mu, \zeta \rangle = df_{X_\zeta} = i_{X_\zeta}\omega,$$

which can be calculated given X_ζ.

1.1 Examples

Most examples focus on the easy to understand case $M = T^*\mathbb{R}^3$ where G is either the group of translations, $\mathbb{R}^3$, the group of rotations, $SO(3)$, or their semi-direct product the Euclidean group in three dimensions. The Islands project[3] has several nice examples including the first one I give here, the case of $SO(3)$ acting on $T^*\mathbb{R}^3$ and some Hyperkähler examples including constructing Taub-Nut space as a Hyperkähler quotient.

Example 1.1. *When $M = T^*\mathbb{R}^3$ and $G = \mathbb{R}^3$ we have the symplectic form*

$$\omega = dx^i \wedge dy_i,$$

[1] If there are any comments or corrections that you feel I should know about you can contact me at cdr1@hw.ac.uk

[2] see `http://wwwf.imperial.ac.uk/~at515/bigworkshop.html` for details of the workshop.

[3] see `http://www.maths.tcd.ie/~islands/index.php?title=Main_Page`

think of the x^i as position coordinates in $\mathbb{R}^3$ and the y_i as the coefficients of a cotangent vector, $y_i dx^i$. The action of an algebra element $\vec{a}$ will generate a vector

$$X = a^i \frac{\partial}{\partial x^i}$$

where the coefficients are constants. Now

$$i_X \omega = a^i dy_i = d(a^i y_i)$$

which tells us that

$$f_a = a^i y_i$$

and since $\vec{a} \in \mathfrak{g}$ we can read off that

$$\mu(\vec{x}, \vec{y}) = \vec{y}.$$

The case of the rotation group proceeds in a similar manner we just need to consider the vector which generates rotations, this will lead to getting a moment for the angular momentum.

Example 1.2. *Another example, taken from [1] this time, is when $M = End(\mathbb{C}^n)$ and $G = U(n)$. In this case the tangent space to N is isomorphic to N with metric*

$$g(A, B) = Re\, \mathrm{tr}(AB^\dagger)$$

and G acts by conjugation. In [1] here he takes

$$f_{X_\zeta} = \frac{i}{2} \mathrm{tr}\left([A, A^\dagger] X_\zeta\right)$$

from which we can read off that the moment map is

$$\mu(A) = \frac{i}{2}[A, A^\dagger].$$

To see this more rigorously consider that an element $\zeta \in \mathfrak{u}(n)$ acts on N through $[X_\zeta, A]$ for X_ζ the associated tangent vector. Then we have that

$$\begin{aligned} \omega([X_\zeta, A]Y) &= g(I[X_\zeta, A]Y), \\ &= Re\ \mathrm{tr}\left(i[X_\zeta, A]Y^\dagger\right), \\ &= (df_{X_\zeta})(Y), \end{aligned}$$

and we can read off μ from f.

Example 1.3. *Our final example is also given in [1] but originates in [3]. Take $M = \mathcal{A}$ the infinite dimensional affine space of connections on a unitary principal bundle over a Riemann surface M. Here the connection is determined by its $(0, 1)$ part, as the $(1, 0)$ part can be found via conjugation, and the tangent space at a connection A is*

$\Omega^{0,1}(M; adP \otimes \mathbb{C})$. *Here* $\mathcal{A}$ *is modelled on* $\Omega^{0,1}(M; adP \otimes \mathbb{C})$. *Now on* $T\mathcal{A}$ *we have the hermitian metric*

$$g(\Psi, \Phi) = 2i \int_M \mathrm{tr}\left(\Psi^\dagger \Phi\right), \tag{1.1}$$

where we are tracing over the adjoint representation, $\dagger$ *is hermitian conjugation and we take the composition* $\Psi^\dagger\Phi$ *to mean the symmetric product* $\star\Psi^\dagger \wedge \Phi$. *There is also a natural symplectic structure given by*

$$\omega(\alpha, \beta) = -2 \int_M \mathrm{tr}\left(\bar{\alpha} \wedge \beta\right)$$

for $\alpha, \beta \in \Omega^{0,1}(M; adP \otimes \mathbb{C})$. *Now for every hermitian metric there is a Kähler form given by* $\omega(X, Y) = g(IX, Y)$ *so we should be able to find an explicit relation between* ω *and* g *in local coordinates, for now I will just satisfy myself that* ω *is definitely symplectic and that it may need to be modified by some constant factors to get that it is the Kähler form.*

We want to consider the action of the group of gauge transformation, $\mathcal{G}$*,endomorphisms of the bundle* P *which has Lie algebra* $\mathfrak{g}$. *Now we can identify* $\mathfrak{g}$ *with* $\Omega^0(M; adP)$*, the space of adjoint valued functions on* M. *This means that we can generate a vector field on* $\mathcal{A}$ *through* $\bar{\partial}_A\zeta$ *for* $\zeta \in \mathfrak{g}$. *Also we should note that the dual of the Lie algebra can be identified with* $\Omega^{1,1}(M; adP \otimes \mathbb{C})$.

This is because we can identify 0*-forms with* 2*-forms through the Hodge star and elements of* $\mathfrak{g}$ *with* $\mathfrak{g}^*$ *through the pairing* $<,>$,

$$\begin{tikzcd}
\Omega^0(M; adP \otimes \mathbb{C}) \arrow[r, leftrightarrow, "\lambda"] \arrow[d, leftrightarrow, "\star"'] & \mathfrak{g} \arrow[d, leftrightarrow, "\langle,\rangle"] \\
\Omega^{1,1}(M; adP \otimes \mathbb{C}) \arrow[r, leftrightarrow, "\tilde{\lambda}"'] & \mathfrak{g}^*
\end{tikzcd}$$

Note that depending on your conventions the Lie algebra of a unitary group, not $\mathfrak{g}$ *but* adP *in this case, will either be hermitian or anti hermitian, for example* $SU(2)$ *has Lie algebra* $\mathfrak{su}(2)$ *which is spanned by either* $-\frac{i}{2}\sigma_i$ *or* $\frac{1}{2}\sigma_i$ *where the first case is anti hermitian, and the second case is hermitian. We will take the Lie algebra to be traceless and anti-hermitian.*

To make things clearer we want to first explore the "real" case, this is based on the discussion in [2]. Here $\mathcal{A}$ *is modelled on* $\Omega^1(M; adP)$ *with* $T_A\mathcal{A} \simeq \Omega^1(M; adP)$ *and the symplectic form will be*

$$\omega(\alpha, \beta) = \int_M \mathrm{tr}\left(\alpha \wedge \beta\right), \qquad \textit{for } \alpha, \beta \in \Omega^1(M; adP).$$

Now we saw that we can identify the adjoint valued functions with $\mathfrak{g}$ *and that we can also generate a vector field from every* $\zeta \in \mathfrak{g}$ *using its infinitesimal action* $d_B\phi$ *for* $d_B \in \mathcal{A}$,

$B \in \Omega^1(M; adP)$. *Consider;*

$$\begin{aligned}
\omega(d_B\zeta, A) &= \int_M \operatorname{tr}\left(d_B\zeta \wedge A\right), \\
&= -\int_M \operatorname{tr}\left(\zeta \wedge d_B A\right), \\
&= -\int_M \operatorname{tr}\left(d_B A \wedge \zeta\right), \\
&= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(\int_M \operatorname{tr}\left(F_{B+\epsilon A} \wedge \zeta\right) - \int_M \operatorname{tr}\left(F_B \wedge \zeta\right)\Big), \\
&= d_{\mathcal{A}}\Big(-\int_M \operatorname{tr}\left(F_B \wedge \zeta\right)\Big)(A), \\
&= d_{\mathcal{A}}\langle \mu(B), \zeta\rangle(A),
\end{aligned}$$

this shows that depending how we interpret the pairing that between $\mathfrak{g}$ and $\mathfrak{g}^*$ we can say that

$$f_\zeta(B) = \langle \mu(B), \zeta\rangle = -\int_M \operatorname{tr}\left(F_B \wedge \zeta\right)$$

is the Hamiltonian function coming from a vector associated to $\zeta \in \mathfrak{g}$ at the connection $d_B \in \mathcal{A}$ and that

$$\mu(B) = F_B$$

is the moment map, where we assume that the pairing $<,>$ involves an integration over M to remove the differential form pieces and get genuine elements of $\mathfrak{g}, \mathfrak{g}^*$.

The complex case will be similar but now we need to be careful about the hermitian metric which we are given as it involves conjugation of both the coordinates and the adjoint values coefficients.

$$g(X, Y) = 2i \int_M \operatorname{tr}\left(X^\dagger Y\right).$$

Also since we are now considering a holomorphic vector bundle the connection is unitary and will be completely determioned by the $0,1$ part as this is related to the $1,0$ part through conjugation. This explains the notation $\bar{\partial}_A$ for the connection. If we take the same starting point as before, $\bar{\partial}_B \in \mathcal{A}, A \in \Omega^{(0,1)}(M; adP \otimes \mathbb{C})$, we have that

$$\begin{aligned}
\omega(\bar{\partial}_B\zeta, A) &= g(I\bar{\partial}_B\zeta, A), \\
&= 2i \int_M (\operatorname{tr}\left((i\bar{\partial}_B\zeta)^\dagger A\right), \\
&= 2\int_M \operatorname{tr}\left(\bar{\partial}_B^\dagger \zeta^\dagger A\right), \\
&= 2\int_M \operatorname{tr}\left(\zeta \partial_B A\right), \\
&= 2\int_M \operatorname{tr}\left(\partial_B A \zeta\right), \\
&= 2 d_{\mathcal{A}}\Big(\int_M \operatorname{tr}\left(F_B \zeta\right)\Big)(A),
\end{aligned}$$

from this we can read off that we have a Hamiltonian function

$$f_\zeta(B) = 2\int_M \mathrm{tr}\,(F_B\zeta)$$

and a moment map

$$\mu_1(B) = F_B.$$

Now we can also consider the action of $\mathcal{G}$ *on* $\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$*. In this case the Kähler metric will be*

$$g(\Lambda,\Psi) = 2i\int_M \mathrm{tr}\left(\Lambda\Psi^\dagger\right), \tag{1.2}$$

where $\Lambda,\Psi\in T_\Phi\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$*. We could proceed by analogy with Example 1.2 and take*

$$f_{X_\zeta}(\Phi) = \int_M \mathrm{tr}\,([\Phi,\Phi^*]\zeta)$$

as a Hamiltonian function at $\Phi\in\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$ *for* $\zeta\in\mathfrak{g}$ *and* $X_\zeta\in\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$ *its associated vector field. Using this we can read off that the moment map is*

$$\mu_2(\Phi) = [\Phi,\Phi^*].$$

However we can also go through the details ourselves starting from the observation that the action of an element $\zeta\in\mathfrak{g}$ *on* $\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$ *is the adjoint action,* $[\Phi,\zeta]$ *for* $\Phi\in\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$*. Using this we consider*

$$\begin{aligned}
\omega([\Phi,\zeta],\Psi) &= g(I[\Phi,\zeta],\Psi),\\
&= 2i\int_M \mathrm{tr}\left(i[\Phi,\zeta]\Psi^\dagger\right),\\
&= -2\int_M \mathrm{tr}\left(\Phi\zeta\Psi^\dagger - \zeta\Phi\Psi^\dagger\right),\\
&= -2\int_M \mathrm{tr}\left(\zeta\Psi^\dagger\Phi - \zeta\Phi\Psi^\dagger\right),\\
&= 2\int_M \mathrm{tr}\left(\zeta[\Phi,\Psi^\dagger]\right),\\
&= 2\int_M \mathrm{tr}\left([\Psi,\Phi^\dagger]\zeta\right),\\
&= \lim_{\varepsilon\to 0}\frac{2}{\varepsilon}\int_M \mathrm{tr}\left([\Phi,(\Phi+\varepsilon\Psi+O(\varepsilon^2))^\dagger]\zeta - [\Phi,\Phi^\dagger]\zeta\right),\\
&= d\Big(2\int_M \mathrm{tr}\left([\Phi,\Phi^\dagger]\zeta\right)\Big)(\Psi),
\end{aligned}$$

where d *is the de-Rham differential for the complex over* $\Omega^{(1,0)}(M; adP\otimes\mathbb{C})$*. This gives us that*

$$f_{[\zeta,\Phi]}(\Phi) = 2\int_M \mathrm{tr}\left([\Phi,\Phi^\dagger]\zeta\right)$$

is a Hamiltonian vector field at $\Phi \in \Omega^{(1,0)}(M; adP \otimes \mathbb{C})$ *and that the moment map is*

$$\mu_2(\Phi) = [\Phi, \Phi^\dagger].$$

This seems a better way to find μ_2 *as we see that the same factor of* 2 *arises in both cases so even if we don't parcel that in with the normalisation of the trace it will be a common factor when we consider the sum of the moment maps. Putting this together we have that*

$$\mu(A, \Phi) = \mu_1(A) + \mu_2(\Phi). \tag{1.3}$$

Building on this last example we have that

$$\mu(A, \Phi) = 0$$

is equivalent to the curvature part of the self duality equations,

$$F_A + [\Phi, \Phi^*] = 0.$$

So this part of the self-duality equations is the moment map for $\mathcal{G}$ acting on the Kähler manifold

$$N = \mathcal{A} \times \Omega^{(1,0)}(M; adP \otimes \mathbb{C}).$$

2 Moduli Space of Solutions to the Self-Duality equations

Changing tactic we now consider the moduli space of solutions to the self-duality equations, in our context this just means the space of connections $A \in \mathcal{A}$ and Higgs fields $\Phi \in \Omega^{(1,0)}(M; adP \otimes \mathbb{C})$ satisfying the self duality equations

$$\bar{\partial}_A \Phi = 0, \tag{2.1}$$

$$F_A + [\Phi, \Phi^*] = 0, \tag{2.2}$$

modulo gauge transformations. Here ad$P$ is a vector bundle over the Riemann surface M and is an associated vector bundle of the principal bundle $P \to \mathbb{R}^4$ that we start with before dimensional reduction, see [1]. I will will not show that $\mathcal{M}$ is smooth but will sketch how to construct this moduli space and explore some of its properties;

1. The self-duality equations can be interpreted as Hyperkähler moment maps, we are all ready part way there after the last section.

2. The space of solutions to the self-duality equations has a Hyperkähler structure.

3. The moduli space, $\mathcal{M}$, can be realised through a Hyperkähler quotient and as such possesses as naturalHyperkähler metric.

4. Finally there is a $\mathbb{CP}^1$'s worth of complex structure, often called a twistor sphere so the space $\mathcal{M} \times \mathbb{CP}^1$ can be identified with the moduli space of stable Higgs bundles, complex structure I, or the moduli space of flat connections, complex structure J.

2.1 Hyperkähler Structure of the Moduli Space

Let us start with the statement that $T_{(A,\Phi)}N$ has a natural Kähler metric coming from the Kähler metrics on the two parts, Equation (1.1) and Equation (1.2), this is

$$g((B_1, \Psi_1), (B_2, \Psi_1)) = 2i \int_M \mathrm{tr}\left(B_1^\dagger B_2 + \Psi_1 \Psi_2^\dagger\right) \tag{2.3}$$

Now the quotient will definitely inherit an inner product which is invariant under gauge transformations and thus a metric. Theorem 6.1 in [1] is the statement that the metric is complete, read geodesically complete. In fact the metric on $\mathcal{M}$ is Hyperkähler, there exists a trio of complex structures, I, J, K which give a representation of the quaternions. Explicitly we have the action of the other complex structures through

$$J(A, B) = (iB^*, -iA^*), \qquad K(A, B) = (-B^*, A^*),$$

note we can show that

$$JK(A, B) = i(A, B) = I(A, B).$$

We also know that for each complex structure there will be a Kähler form and thus a moment map,

$$\omega_1(X, Y) = g(IX, Y), \quad \omega_2(X, Y) = g(JX, Y), \quad \omega_3(X, Y) = g(KX, Y).$$

We know that the moment map corresponding to ω_1 is

$$\mu_1(A, \Phi) = F(A) + [\Phi, \Phi^\dagger],$$

and its zero set gives the second of the self duality equations. The first self duality equation,

$$\bar{\partial}_A \Phi = 0,$$

can also be interpreted as a moment map and this is what we want to do now. N has tangent space $T_{A,\Phi}N \simeq \Omega^{(1,0)}(M; adP \otimes \mathbb{C}) \bigoplus \Omega^{(0,1)}(M; adP \otimes \mathbb{C})$, which has the natural holomorphic symplectic form, $\Omega_I = \omega_J + i\omega_K$, where

$$\Omega_I((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \int_M \mathrm{tr}\left(\Phi_2 \Psi_1 - \Phi_1 \Psi_2\right).$$

Note that in contrast to the symplectic form associated with the metric above this does not have real coefficients so should be thought of as a complex combination of two symplectic forms related to the hermitian metric. It has constant coefficients so is closed over N. The action of an infinitesimal gauge transformation, $\psi \in \Omega^0(M; adP \otimes \mathbb{C})$ on N defines the vector field $X = (\bar{\partial}_A \psi, [\Phi, \psi])$. Now to find the moment map associated to this action we consider,

$$\begin{aligned}
(i_X \omega)(\dot{A}^{0,1}, \dot{\Phi}) &= \omega((d_A'' \psi, [\Phi, \psi]), (\dot{A}^{0,1}, \dot{\Phi})), \\
&= \int_M \mathrm{tr}\left(\dot{\Phi} \bar{\partial}_A \psi - [\Phi, \psi] \dot{A}^{0,1}\right), \\
&= \int_M \mathrm{tr}\left(-\psi \bar{\partial}_A \dot{\Phi} - \psi [\dot{A}^{0,1}, \Phi]\right), \\
&= d_N \Big(- \int_M \mathrm{tr}\left(\bar{\partial}_A \Phi \psi\right) \Big) (\dot{A}^{0,1}, \dot{\Phi}),
\end{aligned}$$

from which we can read off the Hamiltonian function

$$f_X(A,\Phi) = -\int_M \mathrm{tr}\left(\bar{\partial}_A \Phi \psi\right)$$

and the moment map

$$\mu(A,\Phi) = \bar{\partial}_A \Phi$$

whose zero set gives the first self duality equation. We mentioned above that since Ω_I can be decomposed as

$$\Omega_I = \omega_2 + i\omega_3$$

where ω_2, ω_3 are Kähler forms for the other complex structures J and K as above. This means that the self-duality equations are given by

$$\mu_i(A,\Phi) = 0$$

for $i = 1,2,3$ giving the moment maps associated with the action of the group of gauge transformations acting on N with Kähler form ω_i. Now the moduli space, $\mathcal{M}$, of solutions to the self-duality equations was introduced as the space of associated vector bundles, ad$P$ for a given principal bundle P over $\mathbb{R}^4$, over a Riemann surface, M, which have a connection A and a smooth holomorphic section Φ that satisfy the self-duality equations, modulo gauge equivalence,

$$\mathcal{M} = \{(\mathrm{ad}P, M, A, \Phi), A \in \mathcal{A}, \Phi \in \Omega^{(1,0)}(M : adP \otimes \mathbb{C}) | \bar{\partial}_A \Phi = 0, F_A + [\Phi, \Phi^*] = 0\}/\mathcal{G}. \tag{2.4}$$

However, we can also interpret this space as

$$\mathcal{M} = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/\mathcal{G}$$

since the preimage of zero under the μ_i is the set of A, Φ which satisfy the self-duality equations. Since we have see that the μ_i are hyperkähler moment maps, related to the hyperkähler structure on $T_{(A,\Phi)}N$, we can now show that $\mathcal{M}$ also possesses a hyperkähler metric.

Theorem 2.1 (Hitchin Theorem 6.7 [1])**.** *Let M be a compact Riemann surface of genus $g > 1$ and $\mathcal{M}$ the moduli space of irreducible solutions to the $SO(3)$ self duality conditions. Then the natural metric on the $12(g-1)$ dimensional manifold $\mathcal{M}$ is hyperkählerian.*

Proof. Let Y be a tangent vector in N, tangent to the submanifold, $\bar{\mathcal{M}} = \bigcap_{i=1}^{3} \mu_i^{-1}(0)$, of solutions to the self-duality equations. Then we have that

$$\mu_i|_{\bar{\mathcal{M}}} = 0$$

which implies that

$$d\mu_i|_{T_{(A,\Phi)}\bar{\mathcal{M}}} = 0$$

so

$$df_{i,X}(Y) = 0$$

where $f_{i,X}$ is the Hamiltonian function associated to the moment map μ_i and the vector field X generated by the action of $\mathcal{G}$. Now

$$df_{i,X}(Y) = \omega(X, Y)$$

so we have that

$$g(IY, Z) = g(JY, Z) = g(KY, Z) = 0$$

for all vectors Z tangent to the orbit of $\mathcal{G}$, as the kähler forms are invariant under the $\mathcal{G}$ action and the $\mathcal{G}$ action preserves $\bar{\mathcal{M}}$. This gives us that IY, JY, KY are orthogonal to the orbit as well and thus the horizontal space is preserved by I, J, K. Now the action of $\mathcal{G}$ also preserves the complex structures so the tangent space to a point in $\mathcal{M}$ admits an action of the quaternions which is compatible with the metric. This is an almost hyperkähler metric. For it to be hyperkähler we need that the three symplectic forms are closed, [1] has a lemma showing that integrability of the complex three complex structures is equivalent to the closure of their two forms but I will not reiterate that here as it is a standard result. From the projection $p : \bar{\mathcal{M}} \to \mathcal{M}$ we have that

$$p^*\bar{\omega}_i = \omega_i|_{\bar{\mathcal{M}}},$$

where the $\bar{\omega}_i$ are the 2-forms on $\mathcal{M}$ associated to I, J, K. Now $d\omega_i = 0$ so $p^*d\bar{\omega}_i = 0$ and as p is the projection in a fibration, $d\bar{\omega}_i = 0$. Thus the $\bar{\omega}_i$ are all closed and $\mathcal{M}$ has a hyperkähler metric. $\square$

The dimension stated in the theorem and the group given are to make contact with the work earlier in [1], nothing about those facts is used to prove the theorem here.

We have now achieved the first three points, before moving on to the last point we want to make one more observation which singles out the complex structure I. If (A, Φ) is a solution of the self-duality equations then so is $(A, e^{i\theta}\Phi)$ for constant θ. In fact this $U(1)$ action preserves the metric of Equation (2.3) above and thus acts by isometries on $\mathcal{M}$. However, it does not preserve the symplectic form of Equation (2.1) so will only preserve one of the complex structures, I, but not the others, J, K. This action will also have a moment map we we calculate now. The vector field associated with the action is

$$X = (0, i\Phi)$$

so

$$(i_X\omega_1)(Y) = g(IX, Y) = g(-\Phi, Y) = -\frac{1}{2}(dg(\Phi, \Phi))(Y)$$

giving the moment map as

$$-\frac{1}{2}||\Phi||^2_{L^2}.$$

In fact an observation made in Section 7 of [1] is that the complex structure I is one that we get if we interpret $\mathcal{M}$ as the moduli space of stable Higgs bundles, pairs (V, Φ) satisfying the stability condition, through Theorem 4.3 in [1]. However, that is for another talk to make more precise. Also note that the critical points of the moment map coming from the circle action are the fixed points of that action so we could using the moment map as a Morse function to explore the topology of $\mathcal{M}$, this is done in Section 6 of [1]. In fact there it is proven that in this moduli space of stable pairs interpretation $\mathcal{M}$ is non-compact, connected and simply connected. There are also some results about its Betti numbers which I do not quote here.

2.2 The Other Complex Structures

Now that we know that I is special due to its invariance under the circle action it is natural for us to ask about the other complex structures J, K. We can actually do better than just saying that we have three complex structures, for $x \in S^2$, viewed in terms of the embedding coordinates in $\mathbb{R}^3$, we have that

$$(x_1 I + x_2 J + x_3 K)^2 = -1$$

so there is actually a whole S^2's worth of complex structures. This leads us to consider the product space

$$\mathcal{M} \times S^2$$

which, as S^2 is a complex manifold, is a complex manifold with the complex structure (I_x, I_{S^2}), where $I_x = x_1 I + x_2 J + x_3 K$ for $x \in S^2$. This is called the twistor space of complex structures. This space is important in the proof of the following Proposition from [1]

Proposition 2.2 (Proposition 9.1 from [1])**.** *Let $\mathcal{M}$ be the moduli space of solutions to the self-duality equations on a rank-2 vector bundle of odd degree and fixed determinant over a compact Riemann surface of genus at least* 2*. Then*

1. *all the complex structures of the hyperkählerian familly other than $\pm I$ are equivalent,*
2. *With respect to each such structure $\mathcal{M}$ is a Stein manifold,*
3. *$\mathcal{M}$ has no non-constant bounded holomorphic functions.*

I will not prove this hear as it is covered in detail in [1]. The one comment I will make is that the circle action that we encountered above preserves the complex structure of $\mathcal{M} \times S^2$ when we interpret S^2 as the space of covariant constant 2-forms,

$$x_1\omega_1 + x_2\omega_2 + x_3\omega_3,$$

of unit length. As part of the proof in [1] it is shown that the moment map for the circle action becomes a Kähler potential for the Kähler forms ω_2 and ω_3 so they are both cohomologous to zero. We already know that the complex structure I corresponds to the moduli space of stable pairs of Higgs bundles and now want to see what the complex structure J corresponds to. Remember that the tangent space to a point in N with respect to I is

$$T_{(A,\Phi)}N \simeq \Omega^{(0,1)}(M; adP \otimes \mathbb{C}) \bigoplus \Omega^{(1,0)}(M; adP \otimes \mathbb{C}).$$

We can define an isomorphism $\alpha : N \to \mathcal{A} \times \bar{\mathcal{A}}$ by

$$\alpha(A, \Phi) = (\bar{\partial}_A + \Phi^*, \partial_A + \Phi),$$

with $\partial_A + \bar{\partial}_A = d_A$ the covariant derivative with respect to the connection A. The derivative of α is [4]

$$d\alpha(A, B) = (A + B^*, -A^* + B)$$

which composes with J to give

$$d\alpha(J(A, B)) = d\alpha(iB^*, -iA^*) = (iB^* + iA, iB - iA^*) = id\alpha(A, B).$$

So the map α relates N with complex structure J and $\mathcal{A} \times \bar{\mathcal{A}}$ with its natural complex structure multiplication by i. Now we want to know what happens to the self duality equations under this transformation. An element of $\mathcal{A} \times \bar{\mathcal{A}}$ is a pair of connections $(\bar{\partial}_1, \partial_2)$ which combine to give a $PSL(2, \mathbb{C})$ connection $d = \partial_2 + \bar{\partial}_1$. It is in $PSL(2, \mathbb{C})$ as the one form piece will now be a complex linear combination of the two unitary matrix valued one forms in 1 and 2 and $SL(2, \mathbb{C})$ is the complexification of $SU(2)$ so the complex linear combination lives there [5]. Now if we start from a pair (A, Φ) which solve the self duality equations then under α we have that

$$\alpha(A, \Phi) = (\bar{\partial}_A + \Phi^*, \partial_A + \Phi) = (\bar{\partial}_1, \partial_2)$$

so

$$d^2 = (\partial_A + \Phi + \bar{\partial}_A + \Phi^*)^2 = [\partial_A, \bar{\partial}_A] + [\Phi, \Phi^*] = F_A + [\Phi, \Phi^*] = 0$$

so the new connection is flat. Also using ∂_2 and $\bar{\partial}_1$ we get the unitary connections d_1 and d_2 which we met above. Now the operator, $\bar{\partial}_A + \Phi^*$ goes to

$$\partial_A - \Phi$$

under the conjugation denoted by a bar so we have that

$$F_1 = d_1^2 = (\partial_1 + \bar{\partial}_1)^2 = (\bar{\partial}_A + \Phi^* + \partial_A - \Phi)^2 = F_A - [\Phi, \Phi^*] = -2[\Phi, \Phi^*]$$

and

$$F_2 = d_2^2 = (\partial_2 + \bar{\partial}_2)^2 = (\bar{\partial}_A - \Phi^* + \partial_A + \Phi)^2 = F_A - [\Phi, \Phi^*] = -2[\Phi, \Phi^*]$$

giving us that

$$F_1 = F_2.$$

So picking the complex structure J lets us identify solutions of the self-duality equations with $PSL(2, \mathbb{C})$ flat connections. This means that if we choose the complex structure J, $\mathcal{M}$ becomes the moduli space of flat $PSL(2, \mathbb{C})$ connections. In [1] several results about flat $PSL(2, \mathbb{C})$ connections are now presented which are the analogous to results about solutions of the self-duality conditions. These include:

[4] To see this consider that $(d\alpha)(A, B) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\alpha(X + \varepsilon A, Y + \varepsilon B + O(\varepsilon^2)) - \alpha(X, Y)]$

[5] To see why it is $PSL(2, \mathbb{C})$ we just need to remember that we are considering the $SO(3)$ self-duality equations so the quotient by $\pm\mathbb{I}$ has already been done.

1. Theorem 9.13 in [1]; If (A, Φ) is an irreducible solution of the self-duality equations then $\partial_A + \bar{\partial}_A + \Phi + \Phi^*$ is an irreducible, flat, $PSL(2, \mathbb{C})$ connection.

2. Proposition 9.18 in [1]; If we have a pair of irreducible flat $PSL(2, \mathbb{C})$ connections coming from irreducible solutions to the self-duality equations. Then if the flat connections are related by a complex gauge transformation the solutions to the self-duality equations are related by an $SO(3)$ gauge transformation.

3. Theorem 9.19 in [1] where it is attributed to Donaldson; Let P be a prnciple $SO(3)$ bundle over a compact Riemann surface M. For any irreducible flat connection on P^c there is a gauge transformation taking it to a $PSL(2, \mathbb{C})$ connection $A + \psi$ where (A, ψ) satisfy the self-duality equations.

Thus many of the statements that we could prove about $(\mathcal{M}, I)$ can also be proved for $(\mathcal{M}, J)$ though as stated in [1] $(\mathcal{M}, J)$ is really a covering of the space of equivalence classes of flat connections, this is to ensure that the space is smooth and exatly the same thing is done in Section 2 of [1] to get to $(\mathcal{M}, I)$. This will be our stopping point for now but I may come back to these notes as I read more.

References

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